ACCURATE POLYNOMIAL SOLUTIONS FOR BENDING OF PLATES WITH DIFFERENT GEOMETRIES, LOADINGS AND BOUNDARY CONDITIONS

BY

AHMED ABDULLA AL-ALI

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

CIVIL ENGINEERING

DECEMBER 2016

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN- 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **AHMED ABDULLA AL-ALI** under the direction of his thesis advisor and approved by his thesis committee, has been presented and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN CIVIL ENGINEERING**.

Dr. Husain Jubran Al-Gahtani
(Advisor)

Dr. Salah Al-Dulaijan
Department Chairman

Dr. Ali Al-Gadhib
(Member)

Dr. Salam A. Zummo
Dean of Graduate Studies

Dr. Saeid A. Al-Ghamdi
(Member)

15/1/17
Date

Dedication

To my parents,

the reason of what I became today.

To my wife,

thank you for your great support and continuous care.

To my daughter Amna,

the little candle in my life.

To my family and friends,

I am really grateful for all of you.

ACKNOWLEDGMENTS

I would like to express my great love and gratitude to my family, my father and mother, my wife and daughter, for their truthful support and kind patience from the first day of my study. Without their encouragement, I would never overcome the difficulties that I faced in my research.

I am greatly grateful to my supervisor, Dr. Husain Jubran Al-Qahtani, who was like my father from the first day I met him. He always tries to give all the knowledge and help that make my work easier and more efficient. I would like to show him my gratitude and respect for his patience, guidance and directions throughout my research. His honest support and encouragement allowed me to complete this thesis with amazing outcomes in a limited time. He followed and monitored every progress in my research and I have really enjoyed working with him. I am sure that the valuable discussions with him and the attitudes I learned in problem solving will help me a lot in my future research work.

I would also like to thank the committee members of my thesis, Dr. Ali Al-Gadhib and Dr. Saeid Al-Ghamdi, for their assistance and knowledge in the field of mechanics and bending of plates. Their remarks were very important to improve my work and make it more efficient.

I would like to give my sincere thanks to the Civil and Environmental Engineering Department and all the faculty members of the department, especially for Professors in the field of structural engineering who are giving a great support for all the students and providing all the needed help in any stage of the research.

Finally, I would like to acknowledge with thanks to King Fahd University of Petroleum & Minerals (KFUPM), represented by the Deanship of Graduate Studies for providing the opportunity and assistance during my study period.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

$\boldsymbol{a}$	:	Dimension of the plate in the direction of x axis
$\boldsymbol{b}$	:	Dimension of the plate in the direction of y axis
BC	:	Boundary Condition
C	:	Clamped edge
$\boldsymbol{D}$	:	Flexural rigidity of the plate
$\boldsymbol{E}$	:	Young's modulus of the plate material
F	:	Free edge
G	:	Galerkin method
$\boldsymbol{h}$	:	Thickness of the plate
FEM	:	Finite Element Method
FDM	:	Finite Difference Method
$\boldsymbol{M_n}$	:	Bending Moments per unit length of sections of a plate perpendicular to the direction normal to the section
$\boldsymbol{M_x}$	:	Bending Moments per unit length of sections of a plate perpendicular to x axis

$\boldsymbol{M_{xy}}$: Twisting Moments per unit length of sections of a plate perpendicular to x axis

$\boldsymbol{M_{y}}$: Bending Moments per unit length of sections of a plate perpendicular to y axis

MWR : Galerkin Method of Weighted Residuals

$\boldsymbol{m, n}$: Variables used in the deflection equations that affect the size of the deflection equation and the number of terms inside it

PDE : Partial Deferential Equation

$\boldsymbol{\rho}$: Density of the plate material

$\boldsymbol{q}$: Lateral load on the surface of the plate

$\boldsymbol{Q_{x}}$: Shearing Forces parallel to the z axis per unit length of sections of a plate perpendicular to x axis

$\boldsymbol{Q_{y}}$: Shearing Forces parallel to the z axis per unit length of sections of a plate perpendicular to y axis

R : Ritz method

S : Simply Supported edge

$\boldsymbol{v}$: Poisson's ratio of the plate material

$\boldsymbol{V_{x}}$: Kirchhoff Shearing Forces parallel to the z axis of sections of a plate perpendicular to x axis

$\boldsymbol{V_y}$: Kirchhoff Shearing Reactive Forces parallel to the z axis of sections of a plate perpendicular to y axis

$\boldsymbol{w}$: Plate lateral deflection (in the direction of z axis)

ABSTRACT

Full Name : [Ahmed Abdulla Ahmed Ali Hasan Al-Ali]

Thesis Title : [Accurate Polynomial Solutions for Bending of Plates with Different Geometries, loadings and Boundary Conditions]

Major Field : [Civil Engineering]

Date of Degree : [December 2016]

This thesis applies two of the widely known energy methods in analysis to derive approximate polynomials for the solution of bending of thin plates. The derived solutions are more accurate than the available polynomial solutions in literature. The Galerkin-based weighted residual technique and the Ritz method approaches are developed to derive polynomial solutions that are more accurate than most of the famous solutions of basic plates with different geometries, loadings and boundary conditions. Unlike previous polynomial solutions, the new polynomials are capable of exactly satisfying both essential and secondary boundary conditions. The research focuses more on the uniformly loaded rectangular plates, and then shows the ability of applying the same methods to get accurate solutions with other shapes and loadings. The paper develops solutions for cases that are more complicated than the cases discussed in the famous books of bending of plates. For example, all the solutions for the possible 21 boundary conditions combinations for uniformly loaded rectangular plates are derived in this paper. All computations are performed with the help of the symbolic algebraic software, Mathematica. The accuracy of the derived polynomials are compared with the hyperbolic/trigonometric series solutions given by Timoshenko in his book “Theory of

Plates and Shells" [1]. If there is no available solution in Timoshenko for certain plate cases or there is big difference in results, then the solution is checked with results obtained by the finite element software, COMSOL Multiphysics. After stating the conclusions and outcomes, directions and remarks are given for future work.

ملخص الرسالة

الاسم الكامل: أحمد عبدالله أحمد علي حسن العلي

عنوان الرسالة: حلول دقيقة في صورة معادلات متعددة الحدود لانحناء الصفائح مختلفة الأشكال الهندسية و الحمولة والشروط الحدية

التخصص: ماجستير في العلوم - الهندسة المدنية

تاريخ الدرجة العلمية: ديسمبر 2016

تطبق هذه الرسالة اثنين من أشهر الطرق المبنية على معادلات الطاقة والمعروفة من أجل تحليل الصفائح الرقيقة واستخلاص حلول تقريبية لانحناء هذه الصفائح في صورة دوال متعددة الحدود. الحلول المستخلصة أكثر دقة من الحلول المستخلصة في أبحاث سابقة والتي تأتي في صورة دوال متعددة الحدود. يتم تطبيق طريقة تقنية المتبقي الموزون القائمة على طريقة جاليركن وطريقة ريتز لاستخلاص هذه الحلول والتي تعتبر أكثر دقة من معظم الحلول الشهيرة لانحناء الصفائح الرقيقة لمختلف الأشكال الهندسية والأحمال والشروط الحدودية. خلافاً للحلول متعدد الحدود السابقة، الحلول الجديدة المستخرجة في هذه الرسالة قادرة على استيفاء كل الشروط الحدودية. تركز هذه الرسالة بشكل خاص على الصفائح مستطيلة الشكل ذات الحمولة المنتظمة على سطح الصفيحة، ومن ثم تتطرق إلى أنواع أخرى من الصفائح وتبين إمكانية تطبيق نفس الطرق للحصول على حلول دقيقة مع الأشكال الأخرى للصفائح وأنواع الحمولات المختلفة. تطور هذه الأطروحة حلولاً للحالات التي تعتبر أكثر تعقيداً من الحالات التي نوقشت في الكتب الشهيرة في مجال انحناء الصفائح. على سبيل المثال، تستعرض هذه الرسالة حلولاً لكل الـ21 حالة الممكنة للشروط الحدودية للصفائح مستطيلة الشكل ذات الحمولة المنتظمة. يتم تنفيذ كافة العمليات الحسابية في هذا العمل بمساعدة تطبيق الحاسوب وولفرام ماثماتيكا، المختص بحل العمليات الجبرية الرمزية. تتم مقارنة دقة الحلول المشتقة في هذه العمل مع الحلول المقدمة من قبل تيموشينكو التي تأتي في صورة سلاسل قطعية/ مثلثية والتي قدمها في كتابه الشهير "نظرية الصفائح والقشريات". في حالة عدم توفر حل متاح في هذا الكتاب لبعض الحالات، أو لوحظ أن هناك فرقاً كبيراً في النتائج، يتم التحقق من الحلول بمقارنتها مع النتائج التي تم الحصول عليها بواسطة

برنامج العناصر المحدودة، كومسول ملتيفيزيكس. في نهاية الرسالة، وبعد عرض الاستنتاجات المستخلصة من الرسالة، يتم إعطاء بعض التوجيهات والملاحظات التي من الممكن اتباعها قبل الشروع في الدراسات المستقبلية المتعلقة بهذا الموضوع.

CHAPTER 1

INTRODUCTION

1.1 Background

Plates are defined as flat surface structural elements with a thickness that is much smaller than the other two dimensions. Typically, the thickness to width ratio for plates is less than 0.1. Plates usage is not limited to buildings, but they are widely used in many engineering structures such as architectural structures, bridges, hydraulic structures, pavements, containers, airplanes, missiles, ships, instruments, machine parts and even in micro electronic devices.

In General, plates are subjected to perpendicular static or dynamic loads that cause deflections transverse to the plate. There are unlimited geometrical shapes for plates and can take any regular or irregular shape depending on the type of application. Plates' boundaries could be free, clamped, simply supported or mixed, depending on how they are supported.

Plates are usually subdivided based on their thickness and deflection into three main kinds of plates:

1. Thin plate with small deflection, which has small deflection compared to its thickness. For such plate, it is very satisfactory approximation to deal with it in two-dimensional form by assuming:

- There is no deformation in the middle plane of the plate.
- Straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- The Normal stresses in the direction transverse to the plate can be disregarded.

2. Thin plate with large deflection, which has high deflections compared to its thickness. In such cases, the plate will not be bent into a developable surface and that will be accompanied by strain in the middle plane, which will cause stresses that should be considered in deriving the equation of the plate deflection and that makes assumptions used in first type of plates invalid and makes the problem much more complicated.

3. Thick plate, which has relatively big thickness compared to the other two dimensions. For such plate, the approximate theories of thin plates become unreliable, and the problem should be considered as a three-dimensional problem of elasticity.

The present study is limited to the first type of plates, i.e. thin plates undergoing small deflection. It should be noted that there are only limited number of exact solutions available for plate problems, such as uniformly loaded clamped circular and elliptical plates. The existing analytical solutions are in terms of trigonometric and hyperbolic series that satisfy either the boundary conditions or the governing equilibrium equations but not both. For mixed boundary conditions involving clamped and free edges, the solutions involve the superposition of several lengthy series that are too difficult to be derived and programmed.

The aim of this research is to generate accurate polynomial solutions (more accurate than the available solutions) capable of exactly satisfying all the possible boundary conditions and approximately (but fairly accurately) satisfy the equilibrium equations. The polynomial solutions will be obtained for plates with different geometrical shapes, loadings and boundary conditions. The polynomial solutions will be implemented in a WOLFRAM Mathematica software code capable of computing the deflection, the bending moments, shear forces and even stresses at any location of the plate. The developed code will be verified by comparison with the trigonometric/hyperbolic series solutions given in Timoshenko's book [1] as well as with FEM solutions obtained in this research by the use of COMSOL Multiphysics software.

1.2 Literature Review

A brief review of the available analytical solutions of thin plates undergoing small deflection is given below. The first part of literature is about the trigonometric and hyperbolic series solutions which constitutes the major part of the available analytical solutions. The second and third parts are on the literature related to elasticity and energy-based solutions including polynomials.

1.2.1 Hyperbolic Trigonometric Solutions

Most of the research work is based on this type of solutions, which applies Kirchhoff-Love plate theory on certain cases to get out with hyperbolic trigonometric solutions. Following is the summary of the literature review for some references that give that type of solutions.

The “Theory of Plates and Shells, 1959” book written by Timoshenko [1] is perhaps the most famous book on the analysis of plates and shells since it discusses and provides analytical solutions for many shapes of plates with various boundary conditions. The discussed shapes include rectangular, continuous rectangular, triangular, circular, elliptical, sector, skew plates and plates with hole, and each shape has many boundary conditions combinations and loading conditions. The wide number of studied cases, make the book to be the main reference in any plate analysis study. However, due to the complexity of analysis of some cases, there are still some general cases that are missing.

Table 1 Possible combinations of boundary conditions for rectangular plates (Bold discussed by Timoshenko)

Boundary Conditions	**SS**	**SC**	**SF**	**CC**	**CF**	**FF**
SS	**SSSS**	**SSSC**	**SSSF**	**SSCC**	**SSCF**	**SSFF**
SC		**SCSC**	SCSF	**SCCC**	SCCF	SCFF
SF			SFSF	SFCC	SFCF	SFFF
CC				**CCCC**	**CCCF**	CCFF
CF					CFCF	CFFF
FF						**FFFF**

The given solutions for rectangular plates, which have the most number of studied cases in the book, are in the form of hyperbolic trigonometric series. Table 1 shows all the 21 possible combinations of boundary conditions for rectangular plates, where the 11 cases discussed by Timoshenko are indicated in bold.

Chang Fo-van in his study in 1980 [2] gives the deflection solution for the bending of uniformly loaded rectangular cantilever plates by using the idea of generalized simply supported edge together with the method of superposition. The results are very good and give values with error less than 2% when compared to FEM solutions. The shortage in this solution is that it does not give you one deflection equation, but instead it gives 4 very complicated hyperbolic series equations that should be solved together to get the unknown for the studied conditions.

In their study in 2002, Robert L. Taylor and Sanjay Govindjee [3] developed accurate solution to the clamped rectangular plate problem under uniform loading based on the classical double cosine series expansion and an exploitation of the Sherman-Morrison-Woodbury formula. The solution consist of the summation of 3 double cosine series and equate them to q/D which make it very difficult to be solved by hand. They performed numerical solutions using the developed formula for rectangular plates with various side ratios compared them to the solution generated via Hencky's method and the solution showed very similar results and very small error finite element solutions computed using the Bogner-Fox-Schmit element.

In 2007, C.E. İmrak and İ. Gerdemeli [4], in their study on clamped rectangular plates under uniform load, developed a near exact hyperbolic and trigonometric series solution that identically satisfies the boundary conditions on all four edges. The solution has three terms in which the first term corresponds to the case of a strip and the other two terms denote the effects of the edges. They used a quite simple and straightforward method to perform the solution. Compared to some previously developed solutions, their solution showed reasonable agreement.

In his study in 2010, Batista [5] obtained the Fourier series analytical solutions of uniformly loaded rectangular thin plates with symmetrical boundary conditions from the general solution of a biharmonic equation and tabulated the numerical values for all the cases. For cases of plates with two opposite edges simply supported, he obtained well known explicate expressions for unknown coefficients of deflection series expansion and he suggests that this method should be used over the symplectic method for these cases since both methods lead to the same results and among them the Fourier method results are obtained directly from biharmonic equations. However, for the other cases, these coefficients constitute the infinite system of algebraic equations and may be approximately calculated from the truncated system by successive approximations, which appears to converge quickly for the case of the CCCC plate and the FFFF plate and converge slowly for the CCFF plate. Despite that, he still suggests to use this method over the symplectic method because it leads to a solution of the transcendental equation with complex roots and in addition to the solution of infinite system of algebraic equations for unknown eigenvalue expansion coefficients which is not simpler than numerically solving the infinite system of equations obtained by the Fourier method.

1.2.2 Elasticity Solutions

In the literature review for elasticity solutions, the main focus was on the recent solutions using the symplectic elasticity approach.

In his Master of Philosophy thesis in 2007, Cui Shuang [6] presents a bridging analysis for combining the modeling methodology of continuum mechanics/relativity with that of elasticity using a new symplectic approach, which is developed for deriving exact analytical solutions to some basic problems in thin plate deflections that have long been stumbling blocks in the history of elasticity, specifically for plate cases that do not have previously known solutions. The approach employs the Hamiltonian principle with Legendre's transformation and the analytical solutions are obtained by eigenvalue analysis and the expansion of Eigen functions. The new symplectic plate analysis is completely rational and has no trial deflection functions, yet it renders exact solutions beyond the scope of the semi-inverse approaches. By this method, Shuang succeeded to develop solutions for cases not available in Timoshenko's plate theory and other similar theories. For example, he solves for rectangular plates with all possible 21 boundary conditions' combinations. Comparison of the solutions with the available classical solutions shows excellent agreement. One of the limitations of that approach is that it is only applicable to the linear problems (small deflections).

In 2009, Yuemei Liu and Rui Li [7] used the symplectic geometry approach to get accurate bending analysis of rectangular thin plates with two adjacent edges free and the others clamped or simply supported and that presents a breakthrough in solving plate bending problems since they have long been bottlenecks in the history of elasticity. The basic equations for rectangular plates are first transferred into Hamilton canonical equations and using the symplectic approach, the analytical solution of rectangular thin plate is derived. The approach used in the research eliminates the need to pre-determine the deformation function (Unlike the traditional semi-inverse approaches) and is hence more reasonable than conventional methods. Comparison of results with numerical results showed the validity and efficiency of the approach.

1.2.3 Energy-based Trigonometric and Polynomial Solutions

Energy-based solutions are the aim of this research work. So, the main focus in literature review was on this type of solutions, but only few number of studied plate cases were found.

The "Theory of Plates and Shells" book written by Timoshenko [1] was part of the literature review of hyperbolic trigonometric solutions. It appears also in the literature for energy-based solutions since for some plate cases, the solutions where in the form of polynomials based on energy equations. Some of the cases are triangular plates and clamped elliptical plate under uniform loading.

Bhat in his study done in 1985 [8], an orthogonal set of beam characteristic polynomials is generated using the Gram-Schmidt process and is used to determine the plate deflections under static loading in Rayleigh-Ritz method. The first member of the orthogonal polynomial set was constructed as the simplest polynomial that satisfies all the boundary conditions of the corresponding beam problems accompanying the plate problem and the rest of the set was generated using the Gram Schmidt orthogonalization process. The obtained results in the study were for plates with all edges clamped and those with three edges clamped and one edge free for two types of loadings, uniform and hydrostatic. The results found to agree closely with those obtained by previous methods.

Liew in his paper in 1992 [9], presents the $pb-2$ Ritz function to study the static analysis of arbitrarily shaped plates using the principle of minimum potential energy. The $pb-2$ Ritz function consists of the product of a two dimensional polynomial function and a basic function, which is the product of the specified boundary equations, each raised to the power of 0, 1, or 2 corresponding to free, simply supported, or clamped edge. The resulting $pb-2$ Ritz function automatically satisfies the geometric boundary conditions of the plates. The proposed method does not require any discretization, and the solved plate problems demonstrate the accuracy of the method.

In their research in 1998, Saadatpour and Azhari [10] presented a theoretical formulation for the static analysis of arbitrary quadrilateral shaped simply supported plates under uniform loading. Their procedure is based on the Galerkin method and uses the natural coordinates to express the geometry of general plates in a simple form. They programmed the method and several plate examples were solved and showed high accuracy and validity compared to Ritz and FEM.

In 1999, Mbakogu and Pavlovic [11] applied the Galerkin method to the classical bending problem of a uniformly-loaded orthotropic rectangular plate with clamped edges, and used a computer algebra system (Mathematica) to facilitate the tedious and error-prone computations in the assumed deflection function for the plate and get out with a closed-form deflection equation. The accuracy and convergence of the solution depends on the number of approximations (number of terms) as the results compared to series solutions. The given solutions are simple to use and apply for plates with similar conditions. The used method in that research is almost similar to the method that is going to be used in this thesis but for more plate cases.

Osadebe and Aginam in 2011 [12] used an alternative variational approach based on Ritz variational approach (which is based on total potential energy) for the bending analysis of thin isotropic clamped plate. By the formulation, the deformation surface of the clamped plate with uniformly distributed load is approximated to be the sum of products of constructed polynomials in the x and y axes. The constructed polynomials satisfy the all the plate's geometrical boundary conditions in addition to being interdependent and continuous. The sum of product of the constructed polynomial is substituted into plate's differential equations and then solved through minimization principle. Consequently, the deflection equation of the plate is obtained in symbolic analytical form thus enabling the evaluation of deflection at any arbitrary point on the plate. The solution is done for first, second, third and four terms- polynomials and gave improved accuracies compared to the preferred globally (but complicated) Timoshenko's solution.

In his master thesis study in 2011, Balasubramanian [13] used the Galerkin method combined with the help of the symbolic algebraic software, Mathematica to study the deflection of rectangular plates with all edges clamped, triangular plates with all edges clamped and triangular plates with two edges clamped and one edge simply supported with uniformly distributed loading. The lateral deflections of the these plates were expressed in the form of polynomials which satisfy the essential boundary conditions exactly and approximately satisfy the biharmonic equation of the plates everywhere while the secondary boundary conditions are left unsatisfied. The results equations are very simple and can be applied and used easily. When compared with the exact solution carried out with the use of the finite element analysis software, ANSYS, the resulted equations showed good agreement. Due to the limitation of the proposed method, plates with free edges have not been investigated.

In their study done in 2014, Ezeh, Ibearugbulem, Opara and Oguaghamba [14] applied the characteristic orthogonal polynomial to Galerkin indirect variational method for deflection analysis of thin rectangular plates with all edges simply supported to approximate the solution to the partial differential equation of plates. The deflection equation from this study were compared with those of previous researches and the results showed that the average percentage differences recorded for SSSS plates are 0.014% to 0.055%. These differences showed that the shape functions formulated by the characteristic orthogonal polynomial has rapid convergence and is very good approximation of the “exact” displacement functions of the deformed thin rectangular plate under in-plane loading when applied to Galerkin’s buckling load for isotropic plates.

In his own research, Ibearugbulem [15] presented the deflection function for plate analysis in the form product of two mutually perpendicular truncated polynomial series to adopt this function as a very good approximate deflection function for first mode analysis (pure bending, stability, vibration and thermal bending) of plate continuum. He used Ritz energy method as a veritable tool that employs this function for first mode analysis. He found that when the polynomials are truncated at the fifth term, the essential and the secondary boundary conditions are satisfied. Compared to Timoshenko’s solution, his deflection equation for SSSS, CCCC, CSSS, CCSS, CSCS and CCCS rectangular plates under uniform loading showed a maximum difference of 5% which is good enough compared to the simplicity of the equations.

The recent study of Okafor and Udeh [16] is very similar to the study of Ibearugbulem but they compared the solutions for more cases with Timoshenko's solutions for more aspects such as the design factors for deflection and bending of rectangular plates at varying aspect ratio and showed a very small difference.

As seen in the literature for the energy-based trigonometric and polynomial solutions, the previously done studies are mostly on CCCC, SSSS, CSCS, CSSS, SCCC, CCSS, CCCF rectangular plates and CCC and CCS triangular plates. Furthermore, none of the above cases involving free edges was capable of satisfying the secondary boundary conditions. It is the purpose of this study to generate more accurate polynomial that satisfy both the essential and secondary boundary conditions for all cases including plates with free edges.

1.2.4 Closure

It is true and obvious from the literature review that there are so far many numerical methods that have been developed in the past and give accurate solutions, but still there is a big need for simpler analytical solutions. The analytical solutions has many advantages over the numerical ones. They allow us to have a closer view to the stresses and strains variations even after changing the properties and the shape of the plate. Moreover, one can get a good understanding of the physical behavior of the plate under different loadings. The biggest advantage of them is that they provide equations that one can deal with them to get solutions for cases that are more complicated. Also, analytical solutions can be used as an evaluation tool to measure the accuracy of the developed approximate solutions by quantitative comparisons. In this study, new analytical solutions are derived using the idea of energy methods (Galerkin and Ritz methods) to get out with simple and more accurate solutions in the form of series polynomials that can be used for design and optimization purposes. The methods and procedures used are applicable to different kinds of thin plates.

1.3 Significance of Study

Plates have many applications in various science and engineering fields. They can be tiny in size as in electronic devices up to huge sizes such as those used in mega structures, airplanes and spaceships.

As seen in the literature review section, almost all the research work done on plates were done in order to derive analytical solutions that come in the form of hyperbolic and trigonometric series solutions. These solutions become complicated for certain boundary conditions, especially those involving clamped and free edges. Although that there are some famous and widely used solutions, still they are not exact solutions but they give results very close to real deflections and stresses. The applicability of these solutions is questionable because they are very difficult to implement, especially when the shape of the plates is irregular and the boundary conditions are not in a form that make the problem solvable.

Therefore, there is a need for new simple and more accurate analytical solutions that are practical to be used for design and optimization purposes. Therefore, new analytical solutions are derived in this research using the ideas of energy methods.

Furthermore, the available analytical solutions did not consider important boundary conditions such as rectangular plates with the following boundary conditions CFCF, SFCF, CFFF, CCFF. Furthermore, the accuracy of the shear force given by most of available solutions is questionable.

This study produces new accurate solutions for the analysis of thin plates. The obtained solutions are in simple functional forms and more convenient than the existing hyperbolic/trigonometric series solutions for design applications, in general, and for design optimization, in particular. The tasks are achieved through the help of the powerful symbolic software, Mathematica, which has powerful symbolic, numerical and graphical capabilities.

1.4 Scope of Study

The thesis starts with a short review of some of the previously done researches on the deflection of thin plates for various plate shapes and different research and analysis methodologies.

Chapter 2 shows the derivation of the governing equations for the deflection of plates based on the famous Kirchhoff–Love theory of plates. After that, the basic formulations of both Galerkin and Ritz methods are discussed. The general equations that are building the body of the code developed in Mathematica software to generate the solution of a thin plate with any shape, type of loading and boundary conditions for each method are listed. These general equations are then used in the following chapters are used to generate solutions. The last section of this chapter goes over a fast review of COMSOL Multiphysics software and how it can be used to generate finite element method solutions for every case and compare them with the solutions generated by the Galerkin and Ritz methods.

The derived solutions for different plates shapes, boundary conditions and type of loadings are discussed in the following 6 chapters.

Chapters from 3 to 6 establishes solutions for rectangular plates uniformly loaded with stationary vertical load on the surface of the plates, but each chapter discusses a group of different boundary conditions. Chapter 3 discusses the plates cases with two opposite sides simply supported, Chapter 4 discusses the plates cases with two opposite sides clamped, Chapter 5 discusses the plates with two opposite sides unsymmetrical and Chapter 6 discusses the plates supported at corners only.

In Chapter 7, the research focuses on one boundary condition of the rectangular plates, which is the plate simply supported from the all the edges, and derive solutions for different type of loadings to show the ability of applying these methods for loadings other than the uniformly loaded plates.

Similarly, the aim of Chapter 8 is to show the possibility of applying these methods on uniformly loaded plates, but for shapes other than the rectangular plates. Therefore, the solutions of triangular and elliptical plates with different boundary conditions are derived in this chapter.

The total number of cases studied in this research are 28 different cases, the developed codes for each case produce symbolic polynomial solutions that are capable of providing accurate solutions for the plate deflection and all its derivatives including the shear forces. These solutions are utilized to generate detailed tables containing the results of deflection, bending moments and shear forces at the critical locations of the plate and compared with the hyperbolic/ trigonometric series solutions given by Timoshenko in his book “Theory of Plates and Shells” [1] and/ or the solution obtained in this research by the finite element software, COMSOL Multiphysics to verify the accuracy and the validity of the used methods.

Chapter 9 gives a summary of the main outcomes of the research and explores the possible future development and researches related to the analysis of plates deflections based on the applied energy methods.

CHAPTER 2

FUNDAMENTAL FORMULATIONS

2.1 Introduction

This chapter presents the derivation of all the important equations and essential formulations that governs the bending of thin plates based on Kirchhoff–Love theory of plates including the boundary conditions equations. Then it shows the basic idea of the energy methods used in this study; Galerkin and Ritz methods, and explains how they can be applied directly to derive a system of formulas that help us to find accurate solutions for bending of thin plates. These derived formulas represent the core of the current study procedure. At the end, a short overview of Mathematica and COMSOL Multiphysics software is given.

2.2 Governing Equations

A typical differential plate with stress resultants is shown in Figure 1.

The governing differential equation of the plate is based on the famous Kirchhoff–Love theory of plates [1]. To derive the governing equations, assume that we have a plate with load q acting normal to the surface of the plate and the deflection of the plate is small compared to its thickness. Let us consider a small element of that plate and show all the moments and forces acting on the middle plane of the element as shown in Figure 2.

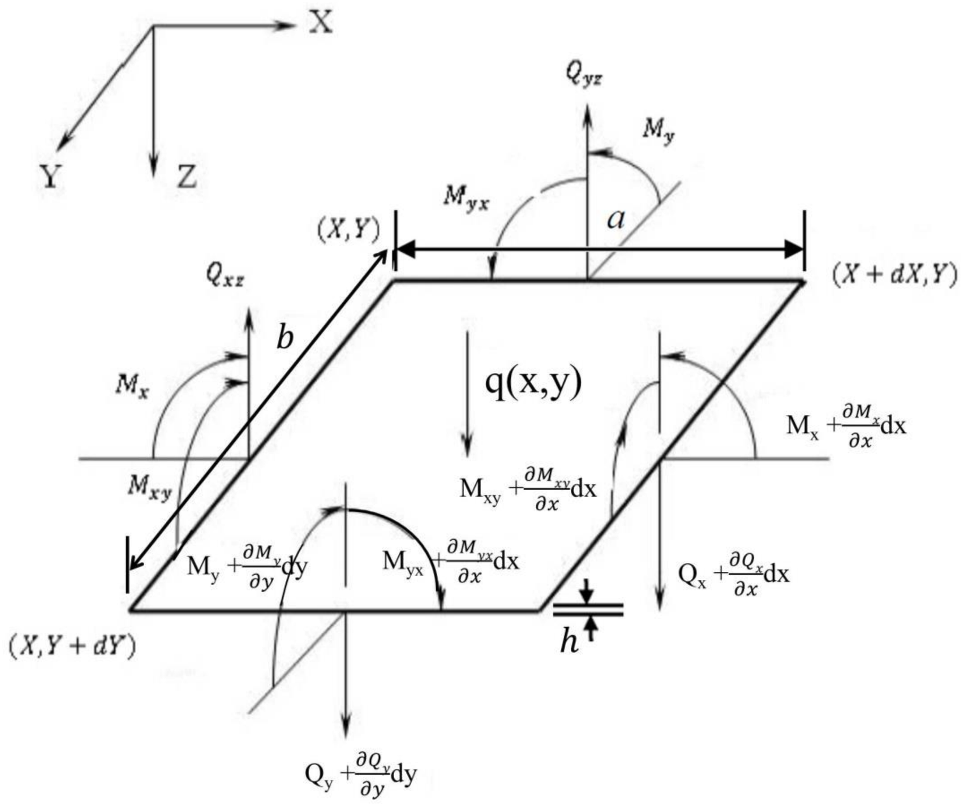

Figure 1 Typical Differential Plate with Dimensions, Edge Moments & Shears

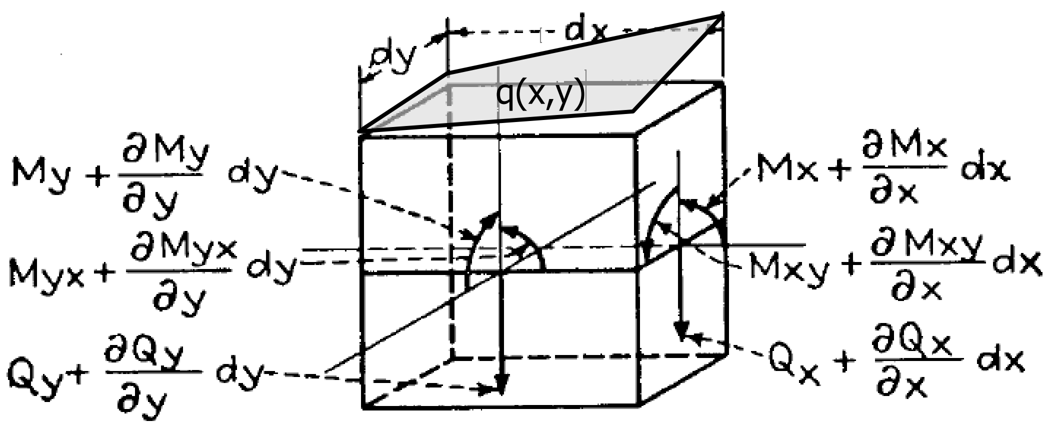

Figure 2 Small Element Cut Out of a Plate Under Load q [1]

In Figure 2, $$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) \tag{2.1}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) \qquad (2.2)$$

$$M_{xy} = -M_{yx} = D(1-v)\frac{\partial^2 w}{\partial x \partial y} \qquad (2.3)$$

Now, let us apply the equations of equilibrium, starting with the summation of forces along the z direction,

$$\frac{\partial Q_x}{\partial x}dxdy + \frac{\partial Q_y}{\partial y}dxdy + qdxdy = 0 \qquad (2.4)$$

and from that we get $$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \qquad (2.5)$$

Taking the moments equilibrium with respect to x axis,

$$\frac{\partial M_{xy}}{\partial x}dxdy - \frac{\partial M_y}{\partial y}dxdy + Q_y dxdy = 0 \qquad (2.6)$$

and from that we get $$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \qquad (2.7)$$

Similarly, by taking the moments equilibrium with respect to y axis,

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \qquad (2.8)$$

And there are no forces in x and y directions and no moments with respect to z axis.

Substitute equations (2.7) & (2.8) in (2.5),

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} = -q \qquad (2.9)$$

And since $M_{xy} = -M_{yx}$, then the equation (2.9) becomes,

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -q \qquad (2.10)$$

Substitute equations (2.1), (2.2) & (2.3) in (2.10), then we get:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (2.11.a)$$

or

$$\nabla^4 w = \frac{q}{D} \qquad (2.11.b)$$

which is the general differential equation of plates, where:

w is the lateral deflection

q is the lateral load

D is the flexural rigidity of the plate, and $D = \frac{Eh^3}{12(1-v^2)}$, where E is Young's modulus of the plate, h is the thickness of the plate and v is Poisson's ratio.

2.3 Boundary Conditions

Clamped Edge Conditions

If a plate is clamped at the boundary, say at edge $x = a$, then the deflection along the edge and the slope of that edge must vanish at the boundary. Therefore, the boundary conditions are:

$$(w)_{x=a} = 0 \qquad \& \qquad (\frac{\partial w}{\partial x})_{x=a} = 0 \qquad (2.12)$$

Similarly if a plate is clamped at the edge $y = b$, then the boundary conditions are:

$$(w)_{y=b} = 0 \qquad \& \qquad (\frac{\partial w}{\partial y})_{y=b} = 0 \qquad (2.13)$$

Simply Supported Edge Conditions

If the plate boundary is prevented from deflecting but allowed to rotate freely around a line along the boundary edge (moment is equal to zero), then the boundary is defined as a simply supported edge. If a plate is simply supported at the edge $x = a$, then the boundary conditions are:

$$(w)_{x=a} = 0 \qquad \& \qquad (M_x)_{x=a} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a} = 0 \tag{2.14}$$

But since the change of w with respect to the y coordinate vanishes along this edge ($\frac{\partial^2 w}{\partial y^2} = 0$), the conditions become:

$$(w)_{x=a} = 0 \qquad \& \qquad (\frac{\partial^2 w}{\partial x^2})_{x=a} = 0 \tag{2.15}$$

Similarly if a plate is simply supported at the edge $y = b$, then the boundary conditions are:

$$(w)_{y=b} = 0 \qquad \& \qquad (\frac{\partial^2 w}{\partial y^2})_{y=b} = 0 \tag{2.16}$$

Free Edge Conditions

If an edge of a plate, say at $x = a$, has no twisting moment, a bending moment or a transverse shear force act on it, then it is a free edge. The boundary conditions are:

$$(M_x)_{x=a} = \left(-D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right)\right)_{x=a} = 0$$

$$\& \qquad (V_x)_{x=a} = \left(-D\left(\frac{\partial^3 w}{\partial x^3} + (2-v)\frac{\partial^3 w}{\partial x \partial y^2}\right)\right)_{x=a} = 0 \tag{2.17}$$

Similarly if a plate is free at the edge $y = b$, then the boundary conditions are:

$$(M_y)_{y=b} = \left(-D\left(\frac{\partial^2 w}{\partial y^2} + v\,\frac{\partial^2 w}{\partial x^2}\right)\right)_{y=b} = 0$$

$$\& \qquad (V_y)_{y=a} = \left(-D\left(\frac{\partial^3 w}{\partial y^3} + (2-v)\,\frac{\partial^3 w}{\partial y \partial x^2}\right)\right)_{y=b} = 0 \qquad (2.18)$$

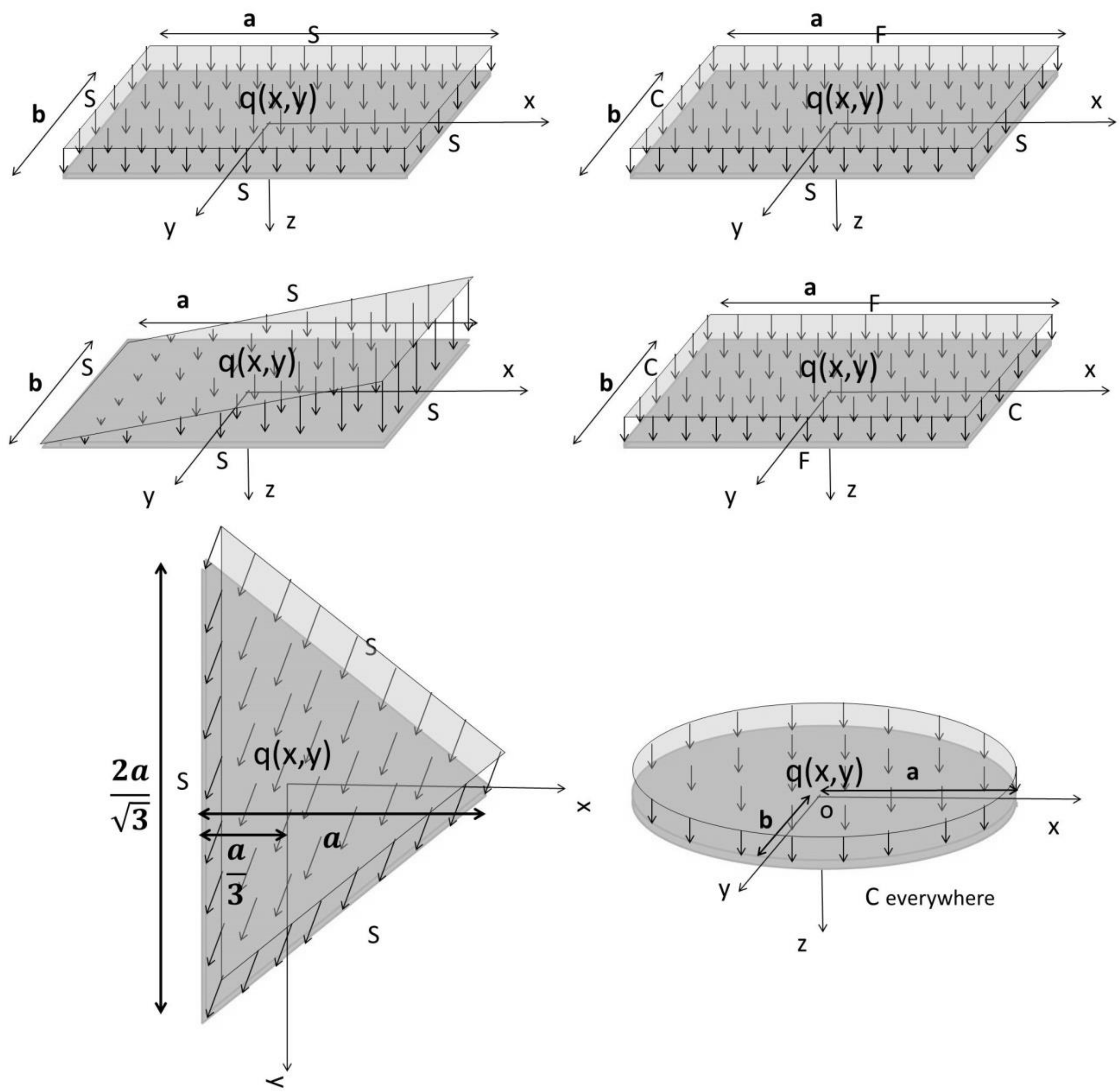

Figure 3 Some Plate Cases Studied in this Thesis (C: clamped edge, S: simply supported edge, F: free edge)

The plates geometries considered in this work include: rectangular, circular, triangular and elliptical shapes. The boundary conditions could be clamped (C), simply supported (S), free (F) or mixed. The loading on plates can take any form of concentrated loads, uniform loading or any load function $q(x, y)$. The study is limited to the analysis of thin plates with small deflection only. The investigation is also limited to static loadings on the surface of the plates. Figure 3 shows representative examples of the cases studied here.

The aim of this research is to derive accurate polynomial solutions for the bending of thin plates with different loadings, geometries and boundary conditions. The symbolic solutions are obtained using the Galerkin-based weighted residual technique and the Ritz method. All computations are performed with the help of the symbolic algebraic software, Mathematica.

2.4 Galerkin Method

Galerkin methods are some of the widely used methods in mathematics; especially in numerical analysis area; to convert a continuous operator problem (example: differential equation) to a discrete problem. These methods were discovered by Walther Ritz, the Swiss mathematician, but it is credited most of the time to the Russian mathematician Boris Galerkin [17].

Galerkin methods are equivalent in principle to the method of variation of parameters when applied to a function space, and then some constraints are applied on the function space to characterize the space with a finite set of basic functions.

Some of the Galerkin methods are:

- The boundary element method for solving integral equations.
- The Galerkin method of weighted residuals (MWR), which is one the popular methods and it is the method used in this research.

MWR methods are used to solve differential equations. The solutions are in the form of a well approximated finite sum of test shape functions ϕ. The basic idea of this method is to evaluate the coefficient value of each corresponding test function. These coefficients have to minimize the error between the total combination of test functions, and the actual real solution. And since the plate deflection equation is in the form of deferential equation, then it can be solved using the MWR method. Related to our work on plates, the Galerkin method should satisfy all the essential (deflection and slope) and secondary (moment and shear) boundary conditions in order to get solutions, which may reduce the possibility of applying this method for all cases as will be shown later. The following paragraphs explain the steps followed in this research in order to derive solutions for plates using the MWR method with the help of Mathematica software.

- The first step is to assume a function $\hat{w}$ as the approximate solution for the deflection of the plate, in the form:

$$\hat{w} = C_j \phi_j(x, y) \quad (2.19)$$

- Next, SolveAlways function in Mathematica is used to apply the boundary conditions to the assumed function $\hat{w}$ to find the numerical relations between the coefficients C_j, that satisfies all boundary conditions. The number of boundary conditions are two at every boundary as explained in section 2.3.

- Since $\nabla^4 w(x, y) = \frac{q(x,y)}{D}$ is the general differential equation of plates, then we set:

$$\nabla^4 \hat{w}(x, y) - \frac{q(x,y)}{D} = R(x, y) \neq 0 \quad (2.20)$$

where R is the residual.

- The parameters C_j are determined by setting the weighted average of the residual over the computational domain to zero

$$\int_\Omega W_i(x, y) R(x, y)\, d\Omega = 0 \quad (2.21)$$

Where i is from 1 to n and $W_i(x, y)$ are weight functions that depend on the type of method employed. In Galerkin method, $W_i = \phi_i$ is used. Where ϕ_i satisfies all boundary conditions as explained in step 2.

- Then the above integral equation becomes:

$$\int_\Omega \phi_i R(x, y)\, d\Omega = 0,\ i = 1, n \quad (2.22)$$

- Substituting the equation of the residual (R) from equation (2.20):

$$\int_{\Omega} \phi_i(\nabla^4 \hat{w}(x,y) - \tfrac{q(x,y)}{D})\, d\Omega = 0,\ i = 1,n \tag{2.23}$$

- Substituting equation (2.19):

$$\int_{\Omega} \phi_i(C_j \nabla^4 \phi_j(x,y) - \tfrac{q(x,y)}{D})\, d\Omega = 0,\ i = 1,n \tag{2.24}$$

- Moving the second term to the other side:

$$\int_{\Omega} \phi_i \nabla^4 \phi_j\, d\Omega C_j = \int_{\Omega} \tfrac{q(x,y)}{D} \phi_i\, d\Omega\,,\ i = 1,n \tag{2.25}$$

Which can be written in the form, $[K]\{C\} = \{F\}$ where:

$$K_{\mathrm{ij}} = \int_{\Omega} \phi_i \nabla^4 \phi_j\, d\Omega \tag{2.26}$$

$$F_i = \int_{\Omega} \phi_i \tfrac{q(x,y)}{D} d\Omega \tag{2.27}$$

The general form of Mathematica code used in this research to derive rectangular plate deflection solutions based on Galerkin method is given in Appendix A.

2.5 Ritz Method

The Ritz method is one of the best direct method to get approximate solutions for boundary value problems. It is named after the Swiss mathematician Walther Ritz. In continuum mechanics; including plate mechanics, any system of particles can be described as energy function, and Ritz method is very effective method to approximate the energy functions in a way that gives the configurations of particles with the least amount of energy. In mathematics, Ritz method is similar to the finite element method used to compute the eigenvectors and eigenvalues of energy functional systems [17].

Ritz method is used to solve differential equations. Exactly as in Galerkin method, the solutions are in the form of a well approximated finite sum of test functions ϕ and the basic idea of the method is to evaluate the coefficient value of each corresponding test function. These coefficients have to minimize the error between the total combination of test functions, and the actual real solution. Related to our work, Ritz method has a big advantage over the Galerkin method since it has only to satisfy the essential (geometric) boundary conditions (i.e. deflection w and slope w′). It does not have to satisfy the other boundary conditions (i.e. moment M and shear V) because they will be automatically (but approximately) satisfied. This advantage makes our solution more flexible which will allow us to use this method in some cases that could not be solved by Galerkin method as will be shown later. This happens because the Ritz method is capable of satisfying the general plate deflection equation exactly without satisfying all the boundary conditions. So, some of the BCs can be neglected in order to get more accurate results that satisfy the general plate deflection equation.

The following paragraphs explain the steps followed in this research in order to derive solutions for plates using the Ritz method with the help of Mathematica software.

- The first step is to assume a function $\hat{w}$ as the approximate solution for the deflection of the plate, in the form:

$$\hat{w} = C_j \phi_j(x, y) \tag{2.28}$$

- Next, SolveAlways function in Mathematica is used to apply the boundary conditions (all boundary conditions can be applied or essential geometric boundary conditions are enough and that reduces the constraints which makes the problem easier to solve) to the assumed function $\hat{w}$ to find the numerical relations between the coefficients C_j, that satisfies the applied boundary conditions. The number of boundary conditions are two at every boundary as explained in section 2.3, but it is enough to just apply the essential BCs.

- The idea in Ritz method to get the values of the parameters C_j, is to use the principle of minimum potential energy Π, which states that Π should be minimized with respect to C_j. Where $\Pi = U - W$, U is the strain energy due to bending and W is work done by external forces. In other words, we should set:

$$\frac{\partial}{\partial C_i}(U - W) = 0, \qquad i = 1, n \tag{2.29}$$

- Lets express U and W in terms of $\hat{w}$. The general equation of Π is:

$$\Pi = \frac{1}{2}\iiint_v (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy})\, dx\, dy\, dz - \iint_A (\rho \hat{w})\, dx\, dy \tag{2.30}$$

After replacing the stresses and strains by their relationships with w, we get:

$$\Pi = \frac{D}{2}\iint_A \left\{ \left(\frac{\partial^2 \hat{w}}{\partial x^2} + \frac{\partial^2 \hat{w}}{\partial y^2} \right)^2 - 2(1 - \nu)\left[\frac{\partial^2 \hat{w}}{\partial x^2}\frac{\partial^2 \hat{w}}{\partial y^2} - \left(\frac{\partial^2 \hat{w}}{\partial x \partial y} \right)^2 \right] \right\} dx\, dy - \iint_A \left(\rho \hat{w} \right) dx\, dy \tag{2.31}$$

So, $U = \frac{D}{2} \iint_A \left\{ \left(\frac{\partial^2 \hat{w}}{\partial x^2} + \frac{\partial^2 \hat{w}}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 \hat{w}}{\partial x^2} \frac{\partial^2 \hat{w}}{\partial y^2} - \left(\frac{\partial^2 \hat{w}}{\partial x \partial y} \right)^2 \right] \right\} dx\, dy$ and

$$W = \iint_A \left(\rho \hat{w} \right) dx\, dy \tag{2.32}$$

- Substituting the equations of U & W in equation (2.29) in indicial notation, substituting $\hat{w}$ from equation (2.28), setting load $\rho = q$ and dividing the whole equation by D give:

$$\frac{1}{2} \frac{\partial}{\partial C_i} \int_\Omega ((C_j \frac{\partial^2 \phi_j}{\partial x^2} + C_j \frac{\partial^2 \phi_j}{\partial y^2})^2 - 2(1-\nu)((C_j \frac{\partial^2 \phi_j}{\partial x^2} C_k \frac{\partial^2 \phi_k}{\partial y^2}) - \left(C_j \frac{\partial^2 \phi_j}{\partial x \partial y} \right)^2))\, d\Omega =$$

$$\frac{\partial}{\partial C_i} \int_\Omega \frac{q}{D} C_j \phi_j \, d\Omega \,, \; i = 1, n \tag{2.33}$$

Further reduction gives:

$$\frac{1}{2} \int_\Omega \left(2 C_j \nabla^2_{\phi_j} \nabla^2_{\phi_i} - 2(1-\nu) \left(\left(C_j \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_i}{\partial y^2} + \frac{\partial^2 \phi_i}{\partial x^2} C_k \frac{\partial^2 \phi_k}{\partial y^2} \right) - 2 C_j \frac{\partial^2 \phi_j}{\partial x \partial y} \frac{\partial^2 \phi_i}{\partial x \partial y} \right) \right) d\Omega =$$

$$\frac{\partial}{\partial C_i} \int_\Omega \frac{q}{D} C_j \phi_j \, d\Omega \,, \qquad i = 1, n \tag{2.34}$$

$$\frac{1}{2} \int_\Omega \left(2 C_j \nabla^2_{\phi_j} \nabla^2_{\phi_i} - 2(1-\nu) \left(\left(C_j \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_i}{\partial y^2} + \frac{\partial^2 \phi_i}{\partial x^2} C_j \frac{\partial^2 \phi_j}{\partial y^2} \right) - 2 C_j \frac{\partial^2 \phi_j}{\partial x \partial y} \frac{\partial^2 \phi_i}{\partial x \partial y} \right) \right) d\Omega =$$

$$\frac{\partial}{\partial C_i} \int_\Omega \frac{q}{D} C_j \phi_j \, d\Omega \,, \qquad i = 1, n \tag{2.35}$$

$$\int_\Omega \left(C_j \nabla^2_{\phi_j} \nabla^2_{\phi_i} - (1-\nu) \left(C_j (\frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_i}{\partial y^2} + \frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial^2 \phi_j}{\partial y^2}) - 2 C_j \frac{\partial^2 \phi_j}{\partial x \partial y} \frac{\partial^2 \phi_i}{\partial x \partial y} \right) \right) d\Omega =$$

$$\int_\Omega \frac{q}{D} \phi_j \, d\Omega \,, \qquad i = 1, n \tag{2.36}$$

$$\int_{\Omega}\left(\nabla^2_{\phi_j}\nabla^2_{\phi_i} - (1-\nu)\left(\frac{\partial^2\phi_j}{\partial x^2}\frac{\partial^2\phi_i}{\partial y^2} + \frac{\partial^2\phi_i}{\partial x^2}\frac{\partial^2\phi_j}{\partial y^2} - 2\frac{\partial^2\phi_j}{\partial x\partial y}\frac{\partial^2\phi_i}{\partial x\partial y}\right)\right)C_j\, d\Omega =$$

$$\int_{\Omega}\frac{q}{D}\phi_j\, d\Omega\,, \qquad i = 1, n \tag{2.37}$$

Which can be written in the form, $[K]\{C\} = \{F\}$ where:

$$K_{\mathrm{ij}} = \int_{\Omega}(\nabla^2_{\phi_j}\nabla^2_{\phi_i} - (1-\nu)(\frac{\partial^2\phi_j}{\partial x^2}\frac{\partial^2\phi_i}{\partial y^2} + \frac{\partial^2\phi_i}{\partial x^2}\frac{\partial^2\phi_j}{\partial y^2} - 2\frac{\partial^2\phi_j}{\partial x\partial y}\frac{\partial^2\phi_i}{\partial x\partial y}))\, d\Omega \tag{2.38}$$

$$F_i = \int_{\Omega}\phi_i\frac{q(x,y)}{2D}\,d\Omega \tag{2.39}$$

The general form of Mathematica code used in this research to derive rectangular plate deflection solutions based on Ritz method is given in Appendix B.

2.6 Wolfram Mathematica

Wolfram Mathematica is the name of the software used to perform all the computations to derive the polynomial solutions for deflection of plates, which is the main outcome of this research. Mathematica is a symbolic algebraic software and Wolfram is the name of the company that produces this software [18].

The wide range of applications that Mathematica can be used for, the ability to deal with symbolic functions in mathematical computations, the powerful and time saving engine, the ability of visualization and plot of problems and the ease of use are the main reasons to choose this computer symbolic software as the main tool of this research.

The software user interface is more like programing or coding software. It has its own coding language which is a very developed knowledge-based language that includes a wide range of programming paradigms and uses its unique concept of symbolic programming to add a new level of flexibility to the concept of programming. Appendix A and Appendix B show examples of the main developed Mathematica codes to get the solutions. The codes include some of the powerful built-in Mathematica functions that helps in deriving solutions.

2.7 COMSOL Multiphysics

COMSOL Multiphysics is a finite element analysis, solver and simulation software. It is a general-purpose software platform, based on advanced numerical methods, for modeling and simulating physics-based problems. It can be used for unlimited number of physics and engineering applications, including coupled phenomena, or multiphysics. One of these applications is the analysis of plates, which is used in this research. In addition to conventional physics-based user interfaces, COMSOL Multiphysics also allows entering coupled systems of partial differential equations (PDEs). The PDEs can be entered directly or using the so-called weak form [19].

As a part of this thesis, all the plates cases that has been studied to derive bending solutions using Galerkin and Ritz methods are also analyzed using COMSOL Multiphysics and tables are derived to compare the results at several points.

As written previously, this software can be used for very complicated cases in various applications, but in this study just a very small part of it has been used. The main 10 steps used in this research to analyze plates are listed in Appendix C.

CHAPTER 3

UNIFORMLY LOADED RECTANGULAR PLATES

WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED

3.1 Introduction

Uniformly loaded rectangular plates having two opposite boundaries simply supported has always been the starting point and the classical case to test in any newly developed method to derive solutions for bending of plates. Hence, bending of these plates is a well-developed subject and wide range of good near accurate solutions for bending of SS plates are available. For that reason, these plates have been chosen to be the first plates to be analyzed in this research and have been given high consideration. Figure 4 shows the configuration and the coordinate system for the plates discussed in this chapter. The plate is simply supported at $x = -a/2$ and $x = a/2$, while the other boundaries at $y = -b/2$ and $y = b/2$ can be either simply supported, clamped or free.

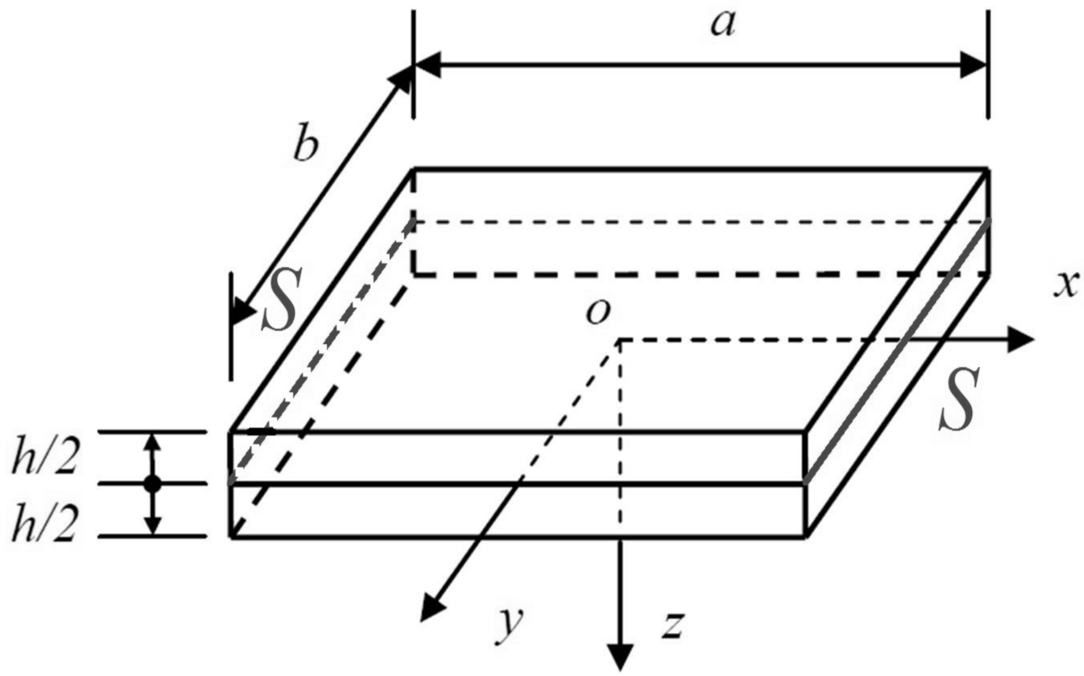

Figure 4 Configuration of Plates with Two Opposite Edges Simply Supported

3.2 Plate Bending Solutions and Comparison of Results

The methodology and the codes used to derive solutions for uniformly loaded rectangular plates with two opposite boundaries simply supported are similar to the steps discussed in sections 2.4 and 2.5, the only few differences are the applied boundary conditions and the assumed starting deflection function (w). Based on that methodology, this section presents the bending solutions for six plates cases with two opposite sides simply supported and compare these results with the solution derived by Timoshenko in his book "Theory of Plates and Shells" [1]. As a part of the methodology used in the study, the values of D (the flexural rigidity of the plate), q (uniform load), a (width of the plate) are always used to be 1 since they are used as scaling parameters. As a result of that, the derived solutions for the deflection, moments and shears of the plates are in the form of functions that are polynomials of x & y multiplied by the scaling parameters$a$, q and D raised to some power. The value of b (length of the plate) is set to be equal to the ratio of b/a and in this chapter cases, Poisson's Ratio (ν) is set to have the value 0.3.

3.2.1 Plate Simply Supported from All the Four Edges (SSSS)

Uniformly loaded fully simply supported plate is the case chosen to be solved first since it is a classical problem and well established solutions are available for comparison. Therefore, more details in the results are given in this section compared to the rest of cases. In this plate case, a convergence study is given and the effect of increasing the number of terms (the power of x and y) on the results is shown.

The plate is simply supported at the boundaries, $x=-a/2$, $x=a/2$, $y=-b/2$ and $y=b/2$. These four boundaries give a total of 8 boundary conditions, 2 at each boundary as discussed in section 2.3, which are:

$$(w)_{x=-a/2}=0 \qquad \& \qquad (M_x)_{x=-a/2}=(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2})_{x=-a/2}=0$$

$$(w)_{x=a/2}=0 \qquad \& \qquad (M_x)_{x=a/2}=(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2})_{x=a/2}=0$$

$$(w)_{y=-b/2}=0 \qquad \& \qquad (M_y)_{y=-b/2}=(\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2})_{y=-b/2}=0$$

$$(w)_{y=b/2}=0 \qquad \& \qquad (M_y)_{y=-b/2}=(\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2})_{y=b/2}=0 \tag{3.1}$$

As mentioned earlier in sections 2.4 and 2.5, to find a solution for the plate using the Galerkin method, all the boundary conditions must be satisfied, while for the Ritz method it is not necessary to satisfy all the boundary conditions, but at least the essential geometry boundary conditions must be satisfied (i.e. deflection and slope). Since SSSS plate is a simple case, it was solved in this study one time with Galerkin method using all the boundary conditions to be satisfied, and two times with Ritz method, the first time using all the boundary conditions to be satisfied and in the second time satisfying only the deflection (w) conditions.

In both methods, the starting assumed function of deflection was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} (a^2 - 4x^2)(b^2 - 4y^2) x^{2i} y^{2j} \qquad (3.2)$$

where $C_{i,j}$ are unknown constants, and n determines the size and the number of terms in the function w. Increasing n, increases the number of terms used (the power of x and y) and therefore improves the results. This function has been specifically chosen because it surely satisfies all the boundary conditions and gives symmetric solution in x and y directions which reduces the time consumed by the software.

As a convergence study, 5 values of n were used, starting from $n = 1$ to $n = 5$, and for each n value, the results at critical points were compared with the solution of Timoshenko [1]. We can see these comparisons in Table 2, Table 3 & Table 4 (values with difference of more than 10% are highlighted with red color):

Table 2 Comparison of results for uniformly loaded rectangular SSSS plate (Galerkin - G & Timoshenko - T) for ($\nu = 0.3$), n from 1 to 4

$n = 1$									
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$Q_{x(-\frac{a}{2},0)}$ $* qa$	$Q_{y(0,-\frac{b}{2})}$ $* qa$	$V_{x(-\frac{a}{2},0)}$ $* qa$	$V_{y(0,-\frac{b}{2})}$ $* qa$	R $* qa^2$
1	G	0.00413703	0.0516301	0.0516301	0.285951	0.285951	0.374914	0.374914	0.0593084
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-1.90%	-7.79%	-7.79%	15.40%	15.40%	10.73%	10.73%	8.76%
1.1	G	0.00496057	0.0594285	0.0536431	0.316427	0.28165	0.404586	0.378625	0.0646498
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-2.28%	-7.27%	-8.81%	12.10%	18.83%	8.05%	13.95%	7.64%
1.2	G	0.00576348	0.0668564	0.0550221	0.344272	0.275623	0.43034	0.378904	0.0688544
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-2.19%	-6.63%	-9.82%	9.40%	21.92%	5.42%	16.36%	6.95%
1.3	G	0.00653131	0.0738309	0.0559111	0.369525	0.268497	0.452631	0.376534	0.0720253
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-2.37%	-6.38%	-11.16%	6.92%	24.79%	3.28%	18.85%	8.83%
1.4	G	0.00725521	0.0803108	0.0564308	0.392315	0.260731	0.471915	0.372171	0.0742934
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-2.91%	-6.37%	-12.41%	4.55%	27.78%	1.27%	20.98%	10.49%
1.5	G	0.00793066	0.0862856	0.0566778	0.412817	0.252653	0.488613	0.366347	0.0757959
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-2.73%	-6.26%	-13.81%	2.64%	30.40%	-0.54%	23.68%	10.83%
3	G	0.013399	0.132918	0.0528814	0.560257	0.156262	0.592272	0.252306	0.0640294
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-9.56%	-11.79%	-30.25%	-13.64%	57.99%	-17.28%	49.34%	31.15%

$n = 2$									
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$Q_{x(-\frac{a}{2},0)}$ $* qa$	$Q_{y(0,-\frac{b}{2})}$ $* qa$	$V_{x(-\frac{a}{2},0)}$ $* qa$	$V_{y(0,-\frac{b}{2})}$ $* qa$	R $* qa^2$
1	G	0.00406221	0.0479133	0.0479133	0.336051	0.336051	0.418136	0.418136	0.0641294
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.05%	-0.03%	-0.03%	0.58%	0.58%	0.44%	0.44%	1.34%
1.1	G	0.00486869	0.0555288	0.0493212	0.35815	0.344309	0.437359	0.435955	0.0700137
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.39%	-0.23%	-0.04%	0.51%	0.78%	0.60%	0.92%	-0.02%
1.2	G	0.0056499	0.0627346	0.0500475	0.37715	0.349808	0.452345	0.449455	0.0748796
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.18%	-0.06%	0.10%	0.75%	0.90%	0.58%	0.78%	-1.19%
1.3	G	0.0063909	0.0694362	0.0502508	0.393441	0.353132	0.463893	0.459367	0.0788173
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.17%	-0.05%	0.10%	0.90%	1.08%	0.88%	1.00%	0.23%
1.4	G	0.00708253	0.0755871	0.0500664	0.407343	0.354708	0.472641	0.466292	0.0819462
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.46%	-0.12%	0.27%	0.89%	1.74%	1.12%	1.00%	1.27%
1.5	G	0.0077201	0.0811769	0.049605	0.419096	0.354848	0.479057	0.470713	0.0843934
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	0.00%	0.03%	0.39%	1.16%	2.25%	1.43%	1.93%	0.71%
3	G	0.012108	0.117279	0.0380961	0.471378	0.294275	0.477559	0.421723	0.0881465
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	1.00%	1.36%	6.17%	4.39%	20.89%	5.43%	15.32%	5.22%

$n = 3$									
b/a	**Method**	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$Q_{x(-\frac{a}{2},0)} * qa$	$Q_{y(0,-\frac{b}{2})} * qa$	$V_{x(-\frac{a}{2},0)} * qa$	$V_{y(0,-\frac{b}{2})} * qa$	$R * qa^2$
1	G	0.00406228	0.0478746	0.0478746	0.338538	0.338538	0.42153	0.42153	0.0646868
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.06%	0.05%	0.05%	-0.16%	-0.16%	-0.36%	-0.36%	0.48%
1.1	G	0.00486887	0.0554701	0.0493026	0.360496	0.347358	0.440879	0.439647	0.0706193
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.39%	-0.13%	-0.01%	-0.14%	-0.10%	-0.20%	0.08%	-0.88%
1.2	G	0.00565056	0.0626911	0.0500847	0.381766	0.355299	0.458594	0.455548	0.0755947
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.19%	0.01%	0.03%	-0.46%	-0.65%	-0.79%	-0.56%	-2.16%
1.3	G	0.00639196	0.0693422	0.0502969	0.395391	0.357806	0.467423	0.463895	0.0795083
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.19%	0.08%	0.01%	0.41%	-0.23%	0.12%	0.02%	-0.64%
1.4	G	0.00708438	0.0754692	0.0501503	0.409616	0.360359	0.476826	0.471362	0.0827576
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.49%	0.04%	0.10%	0.34%	0.18%	0.25%	-0.08%	0.29%
1.5	G	0.00772388	0.0811081	0.0498092	0.419728	0.365102	0.48146	0.480286	0.0851534
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-0.05%	0.11%	-0.02%	1.01%	-0.58%	0.93%	-0.06%	-0.18%
3	G	0.0122409	0.118872	0.040862	0.498603	0.363529	0.51185	0.493959	0.0930937
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-0.09%	0.02%	-0.65%	-1.14%	2.28%	-1.36%	0.81%	-0.10%
$n = 4$									
b/a	**Method**	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$Q_{x(-\frac{a}{2},0)} * qa$	$Q_{y(0,-\frac{b}{2})} * qa$	$V_{x(-\frac{a}{2},0)} * qa$	$V_{y(0,-\frac{b}{2})} * qa$	$R * qa^2$
1	G	0.00406241	0.0479015	0.0479015	0.335817	0.335817	0.418501	0.418501	0.0648128
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.06%	0.00%	0.00%	0.65%	0.65%	0.36%	0.36%	0.29%
1.1	G	0.00486908	0.0555195	0.0493401	0.363075	0.34967	0.443874	0.442182	0.0707619
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.39%	-0.22%	-0.08%	-0.85%	-0.77%	-0.88%	-0.50%	-1.09%
1.2	G	0.00565069	0.0627112	0.0501054	0.379026	0.355802	0.455431	0.456201	0.0757801
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.19%	-0.02%	-0.01%	0.26%	-0.79%	-0.09%	-0.71%	-2.41%
1.3	G	0.00639258	0.0694217	0.0504116	0.399841	0.354467	0.471433	0.461384	0.0796492
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.20%	-0.03%	-0.22%	-0.72%	0.71%	-0.73%	0.56%	-0.82%
1.4	G	0.00708445	0.075488	0.0501603	0.417072	0.364652	0.484831	0.476123	0.0826304
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.49%	0.02%	0.08%	-1.48%	-1.01%	-1.43%	-1.09%	0.45%
1.5	G	0.0077213	0.0808646	0.0495223	0.435491	0.372874	0.500274	0.487969	0.0849504
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-0.02%	0.41%	0.56%	-2.71%	-2.72%	-2.94%	-1.66%	0.06%
3	G	0.0122299	0.118669	0.0404861	0.49696	0.381815	0.509276	0.512703	0.0935625
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	0.00%	0.19%	0.28%	-0.80%	-2.64%	-0.85%	-2.95%	-0.60%

Table 3 Comparison of results for uniformly loaded rectangular SSSS plate (Ritz 1 – R1 [All BCs] & Timoshenko - T) for ($\nu = 0.3$), $n = 1, 3$ & 5

b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$Q_{x(-\frac{a}{2},0)} * qa$	$Q_{y(0,-\frac{b}{2})} * qa$	$V_{x(-\frac{a}{2},0)} * qa$	$V_{y(0,-\frac{b}{2})} * qa$	$R * qa^2$
$n = 1$									
1	R1	0.00413702	0.05163	0.05163	0.285951	0.285951	0.374914	0.374914	0.0593084
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-1.90%	-7.79%	-7.79%	15.40%	15.40%	10.73%	10.73%	8.76%
1.1	R1	0.00496057	0.0594285	0.0536431	0.316427	0.28165	0.404586	0.378625	0.0646498
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-2.28%	-7.27%	-8.81%	12.10%	18.83%	8.05%	13.95%	7.64%
1.2	R1	0.00576348	0.0668564	0.0550221	0.344272	0.275623	0.43034	0.378904	0.0688544
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-2.19%	-6.63%	-9.82%	9.40%	21.92%	5.42%	16.36%	6.95%
1.3	R1	0.00653131	0.0738309	0.0559111	0.369525	0.268497	0.452631	0.376534	0.0720253
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-2.37%	-6.38%	-11.16%	6.92%	24.79%	3.28%	18.85%	8.83%
1.4	R1	0.00725522	0.0803108	0.0564308	0.392315	0.260731	0.471915	0.372171	0.0742934
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-2.91%	-6.37%	-12.41%	4.55%	27.78%	1.27%	20.98%	10.49%
1.5	R1	0.00793066	0.0862856	0.0566778	0.412817	0.252653	0.488613	0.366347	0.0757959
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-2.73%	-6.26%	-13.81%	2.64%	30.40%	-0.54%	23.68%	10.83%
3	R1	0.013399	0.132918	0.0528814	0.560257	0.156262	0.592272	0.252306	0.0640294
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-9.56%	-11.79%	-30.25%	-13.64%	57.99%	-17.28%	49.34%	31.15%
$n = 3$									
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$Q_{x(-\frac{a}{2},0)} * qa$	$Q_{y(0,-\frac{b}{2})} * qa$	$V_{x(-\frac{a}{2},0)} * qa$	$V_{y(0,-\frac{b}{2})} * qa$	$R * qa^2$
1	R1	0.00406205	0.047828	0.047828	0.335628	0.335628	0.418291	0.418291	0.0646303
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.05%	0.15%	0.15%	0.70%	0.70%	0.41%	0.41%	0.57%
1.1	R1	0.00486869	0.0554402	0.0492791	0.358912	0.346929	0.439209	0.439242	0.0706294
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.39%	-0.07%	0.04%	0.30%	0.02%	0.18%	0.17%	-0.90%
1.2	R1	0.00564998	0.0625756	0.0499859	0.373672	0.348953	0.449658	0.448032	0.0753348
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.18%	0.20%	0.23%	1.67%	1.15%	1.17%	1.10%	-1.80%
1.3	R1	0.00639175	0.0693029	0.0502692	0.392539	0.356687	0.464344	0.46257	0.0794535
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.18%	0.14%	0.06%	1.12%	0.09%	0.78%	0.31%	-0.57%
1.4	R1	0.00708452	0.07548	0.0501655	0.408616	0.361318	0.475666	0.472428	0.0827055
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.49%	0.03%	0.07%	0.58%	-0.09%	0.49%	-0.30%	0.35%
1.5	R1	0.00772391	0.0811359	0.0498238	0.424137	0.366384	0.486306	0.482035	0.0853272
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-0.05%	0.08%	-0.05%	-0.03%	-0.93%	-0.06%	-0.42%	-0.38%
3	R1	0.0122349	0.118029	0.0404829	0.455679	0.351088	0.468316	0.475805	0.0925279
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-0.04%	0.73%	0.29%	7.57%	5.62%	7.26%	4.46%	0.51%

$n = 5$									
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$Q_{x(-\frac{a}{2},0)}$ $* qa$	$Q_{y(0,-\frac{b}{2})}$ $* qa$	$V_{x(-\frac{a}{2},0)}$ $* qa$	$V_{y(0,-\frac{b}{2})}$ $* qa$	R $* qa^2$
1	R1	0.00406127	0.0476689	0.0476689	0.351897	0.351897	0.43548	0.43548	0.0645957
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.03%	0.48%	0.48%	-4.11%	-4.11%	-3.69%	-3.69%	0.62%
1.1	R1	0.00486805	0.0553544	0.0491469	0.368611	0.356318	0.449562	0.448573	0.0703776
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.37%	0.08%	0.31%	-2.39%	-2.69%	-2.17%	-1.95%	-0.54%
1.2	R1	0.00565066	0.0626831	0.0501122	0.384583	0.349399	0.461273	0.449851	0.07574
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.19%	0.03%	-0.02%	-1.21%	1.02%	-1.38%	0.70%	-2.35%
1.3	R1	0.00639167	0.0692941	0.0502827	0.411616	0.364988	0.484068	0.472283	0.0797667
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.18%	0.15%	0.03%	-3.68%	-2.24%	-3.43%	-1.79%	-0.97%
1.4	R1	0.0070851	0.0756025	0.0502279	0.406691	0.360041	0.473812	0.471438	0.082993
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.50%	-0.14%	-0.06%	1.05%	0.27%	0.88%	-0.09%	0.01%
1.5	R1	0.00772375	0.0811148	0.0498132	0.429905	0.365291	0.492108	0.480917	0.0856053
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-0.05%	0.10%	-0.03%	-1.39%	-0.63%	-1.26%	-0.19%	-0.71%
3	R1	0.0122332	0.118816	0.0406462	0.471096	0.370051	0.482228	0.49896	0.0941702
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-0.03%	0.07%	-0.11%	4.44%	0.52%	4.51%	-0.19%	-1.26%

Table 4 Comparison of results for uniformly loaded rectangular SSSS plate (Ritz 2 – R2 [only w BCs] & Timoshenko - T) for ($\nu = 0.3$), n from 1 to 5

$n = 1$									
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$Q_{x(-\frac{a}{2},0)}$ $* qa$	$Q_{y(0,-\frac{b}{2})}$ $* qa$	$V_{x(-\frac{a}{2},0)}$ $* qa$	$V_{y(0,-\frac{b}{2})}$ $* qa$	R $* qa^2$
1.1	R2	0.00491287	0.0577231	0.0517478	0.289528	0.252968	0.373433	0.348269	0.0718031
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-1.30%	-4.19%	-4.97%	19.58%	27.10%	15.13%	20.85%	-2.58%
1.2	R2	0.0057039	0.0650303	0.0528121	0.317333	0.245482	0.398009	0.348602	0.0768413
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-1.13%	-3.72%	-5.41%	16.49%	30.46%	12.53%	23.05%	-3.84%
1.3	R2	0.00645658	0.0718444	0.0533544	0.342564	0.23752	0.419268	0.347148	0.0809709
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-1.20%	-3.52%	-6.07%	13.71%	33.47%	10.41%	25.18%	-2.49%
1.4	R2	0.00716175	0.0781211	0.0535015	0.365231	0.229561	0.437512	0.344569	0.0843209
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-1.59%	-3.47%	-6.58%	11.14%	36.41%	8.47%	26.84%	-1.59%
1.5	R2	0.00781459	0.0838481	0.0533557	0.385436	0.221923	0.453065	0.341371	0.087024
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-1.23%	-3.26%	-7.14%	9.10%	38.86%	6.78%	28.88%	-2.38%
3	R2	0.0124756	0.122114	0.0435551	0.50387	0.178757	0.520111	0.323397	0.100193
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-2.01%	-2.70%	-7.28%	-2.20%	51.95%	-2.99%	35.06%	-7.73%

b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$Q_{x(-\frac{a}{2},0)} * qa$	$Q_{y(0,-\frac{b}{2})} * qa$	$V_{x(-\frac{a}{2},0)} * qa$	$V_{y(0,-\frac{b}{2})} * qa$	$R * qa^2$
$n = 2$									
1	R2	0.00406218	0.0478917	0.0478917	0.337128	0.337128	0.419432	0.419432	0.0651536
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.05%	0.02%	0.02%	0.26%	0.26%	0.14%	0.14%	-0.24%
1.1	R2	0.00486868	0.0554999	0.0493039	0.359886	0.344601	0.439534	0.436228	0.0711451
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.39%	-0.18%	-0.01%	0.03%	0.69%	0.11%	0.86%	-1.64%
1.2	R2	0.00564998	0.062699	0.050037	0.379412	0.349037	0.455233	0.448396	0.0761279
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.18%	0.00%	0.13%	0.15%	1.12%	-0.05%	1.02%	-2.88%
1.3	R2	0.00639118	0.0693963	0.0502518	0.396117	0.350976	0.467351	0.456621	0.0801978
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.18%	0.01%	0.10%	0.22%	1.69%	0.14%	1.59%	-1.52%
1.4	R2	0.00708314	0.0755455	0.0500838	0.410384	0.350878	0.476596	0.46155	0.0834758
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.47%	-0.06%	0.23%	0.15%	2.80%	0.29%	2.01%	-0.57%
1.5	R2	0.0077212	0.0811338	0.0496422	0.422554	0.349126	0.483557	0.463767	0.0860869
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-0.02%	0.08%	0.32%	0.34%	3.82%	0.50%	3.38%	-1.28%
3	R2	0.0121619	0.117733	0.0390032	0.485951	0.257439	0.495014	0.385665	0.0952536
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	0.56%	0.98%	3.93%	1.43%	30.80%	1.98%	22.56%	-2.42%
$n = 3$									
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$Q_{x(-\frac{a}{2},0)} * qa$	$Q_{y(0,-\frac{b}{2})} * qa$	$V_{x(-\frac{a}{2},0)} * qa$	$V_{y(0,-\frac{b}{2})} * qa$	$R * qa^2$
1	R2	0.00406233	0.0478834	0.0478834	0.339158	0.339158	0.422131	0.422131	0.0650293
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.06%	0.03%	0.03%	-0.34%	-0.34%	-0.51%	-0.51%	-0.05%
1.1	R2	0.00486893	0.0554817	0.0493132	0.361352	0.348231	0.441715	0.440583	0.0710102
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.39%	-0.15%	-0.03%	-0.38%	-0.35%	-0.39%	-0.13%	-1.44%
1.2	R2	0.00565049	0.0626783	0.0500747	0.38089	0.35517	0.457497	0.455344	0.0759857
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.19%	0.03%	0.05%	-0.23%	-0.61%	-0.55%	-0.52%	-2.68%
1.3	R2	0.00639216	0.0693806	0.0503291	0.397965	0.360488	0.470075	0.467068	0.0800512
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.19%	0.03%	-0.06%	-0.24%	-0.98%	-0.44%	-0.66%	-1.33%
1.4	R2	0.00708486	0.0755422	0.0502118	0.412791	0.36456	0.479985	0.476312	0.0833266
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.49%	-0.06%	-0.02%	-0.44%	-0.99%	-0.42%	-1.13%	-0.39%
1.5	R2	0.00772396	0.0811507	0.049832	0.425604	0.367658	0.487705	0.483539	0.0859364
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-0.05%	0.06%	-0.06%	-0.38%	-1.28%	-0.35%	-0.74%	-1.10%
3	R2	0.0122393	0.118935	0.0408421	0.494191	0.356877	0.506815	0.48633	0.0949716
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-0.08%	-0.03%	-0.60%	-0.24%	4.07%	-0.36%	2.34%	-2.12%

$n = 4$									
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$Q_{x(-\frac{a}{2},0)}$ $* qa$	$Q_{y(0,-\frac{b}{2})}$ $* qa$	$V_{x(-\frac{a}{2},0)}$ $* qa$	$V_{y(0,-\frac{b}{2})}$ $* qa$	R $* qa^2$
1	R2	0.00406234	0.0478823	0.0478823	0.338582	0.338582	0.4213	0.4213	0.0649655
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.06%	0.04%	0.04%	-0.17%	-0.17%	-0.31%	-0.31%	0.05%
1.1	R2	0.00486896	0.0554848	0.049318	0.3593	0.346136	0.439427	0.438252	0.0709705
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.39%	-0.15%	-0.04%	0.19%	0.25%	0.13%	0.40%	-1.39%
1.2	R2	0.0056505	0.0626706	0.0500714	0.381674	0.355602	0.457981	0.455394	0.075885
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.19%	0.05%	0.06%	-0.44%	-0.74%	-0.66%	-0.53%	-2.55%
1.3	R2	0.00639221	0.0693855	0.050338	0.395611	0.357864	0.467439	0.464147	0.0800076
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.19%	0.02%	-0.08%	0.35%	-0.24%	0.12%	-0.03%	-1.28%
1.4	R2	0.00708492	0.0755496	0.0502231	0.410131	0.361575	0.477018	0.472997	0.0832846
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.50%	-0.07%	-0.05%	0.21%	-0.16%	0.21%	-0.42%	-0.34%
1.5	R2	0.00772403	0.0811609	0.0498449	0.422707	0.364439	0.484477	0.479953	0.0858955
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-0.05%	0.05%	-0.09%	0.30%	-0.40%	0.31%	0.01%	-1.05%
3	R2	0.0122327	0.118881	0.0406302	0.49202	0.377216	0.503851	0.506907	0.0948933
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-0.02%	0.02%	-0.07%	0.20%	-1.40%	0.23%	-1.79%	-2.04%
$n = 5$									
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$Q_{x(-\frac{a}{2},0)}$ $* qa$	$Q_{y(0,-\frac{b}{2})}$ $* qa$	$V_{x(-\frac{a}{2},0)}$ $* qa$	$V_{y(0,-\frac{b}{2})}$ $* qa$	R $* qa^2$
1	R2	0.00406316	0.0482246	0.0482246	0.369747	0.369747	0.454589	0.454589	0.062611
	T	0.00406	0.0479	0.0479	0.338	0.338	0.420	0.420	0.065
	% Diff.	-0.08%	-0.68%	-0.68%	-9.39%	-9.39%	-8.24%	-8.24%	3.68%
1.1	R2	0.00486896	0.0554839	0.0493168	0.35996	0.346134	0.440185	0.438349	0.0709503
	T	0.00485	0.0554	0.0493	0.36	0.347	0.440	0.440	0.07
	% Diff.	-0.39%	-0.15%	-0.03%	0.01%	0.25%	-0.04%	0.38%	-1.36%
1.2	R2	0.00565053	0.0626819	0.0500817	0.378811	0.353364	0.455272	0.453384	0.0759197
	T	0.00564	0.0627	0.0501	0.38	0.353	0.455	0.453	0.074
	% Diff.	-0.19%	0.03%	0.04%	0.31%	-0.10%	-0.06%	-0.08%	-2.59%
1.3	R2	0.00639221	0.0693868	0.0503382	0.396903	0.357423	0.468854	0.463856	0.0799733
	T	0.00638	0.0694	0.0503	0.397	0.357	0.468	0.464	0.079
	% Diff.	-0.19%	0.02%	-0.08%	0.02%	-0.12%	-0.18%	0.03%	-1.23%
1.4	R2	0.00708492	0.0755491	0.0502218	0.410991	0.361234	0.478009	0.472793	0.0832579
	T	0.00705	0.0755	0.0502	0.411	0.361	0.478	0.471	0.083
	% Diff.	-0.50%	-0.07%	-0.04%	0.00%	-0.06%	0.00%	-0.38%	-0.31%
1.5	R2	0.00772402	0.0811588	0.049842	0.423572	0.363748	0.485483	0.479415	0.085869
	T	0.00772	0.0812	0.0498	0.424	0.363	0.486	0.480	0.085
	% Diff.	-0.05%	0.05%	-0.08%	0.10%	-0.21%	0.11%	0.12%	-1.02%
3	R2	0.0122327	0.118856	0.0406206	0.493864	0.37406	0.505958	0.503829	0.0948595
	T	0.01223	0.1189	0.0406	0.493	0.372	0.505	0.498	0.093
	% Diff.	-0.02%	0.04%	-0.05%	-0.18%	-0.55%	-0.19%	-1.17%	-2.00%

By taking a look over the previous three tables, the first obvious conclusion from the comparison that as n increases (i.e. number of terms and power of x and y in equations increase), the error in the results compared with Timoshenko's solution becomes smaller. For $n = 1$, in all the tables, there is error difference as high as 60% in some values but as n reaches 3 we see that the maximum error is < 10% and most of the values has an error of less than 1%. For n larger than 3, the values perfectly match.

The second remark is related to the b/a ratio. It is clear from comparison that the error increases a little bit as b/a ratio increases, which tells us that higher number of terms (n) is needed for higher ratios.

The third remark requires comparison between the results of the methods used in this research. On the first hand, deep comparison between results of the Galerkin and the first results of the Ritz method (by satisfying all the BCs) (Table 2 & Table 3) shows almost no difference in results, so they had the same accuracy. On the hand, the second results of the Ritz method (when only the essential BCs were satisfied) (Table 4) shows higher accuracy when compared to Timoshenko's solution and totally agrees with it even at values of n as small as 3. This remark is very important and leads to a general conclusion at the end of this thesis, which states that for complicated plate cases, only the Ritz method is able to find accurate solutions for such cases. In order to get more accurate results using the Ritz method, some of the BCs (secondary BCs like moments and shears) might be neglected to make the problem more flexible and able to exactly satisfy the general equation of plate deflection. However, using the Galerkin method or in the case of applying all the BCs in the Ritz method makes the problem more constrained and sometimes not able to satisfy the plate general differential equation.

The convergence study is only shown in this plate case since it is a main case and in order to show the procedure followed in the remaining cases to decide the proper n value for each case. In all the following cases, comparison tables for values of n starting of 1 were developed but only tables for one n value is shown which is the lowest n value that gives accurate enough results.

As the main outcome of analysis of the plates in this research, a polynomial deflection equation is derived for every v, n and b/a ratio values. These equations are becoming very long as n value increases. It is not possible to list all the derived equations in this thesis, however, the Galerkin method equations derived for this plate case with b/a ratio $=$ 1 and Poisson's ratio $v = 0.3$ are listed in Appendix D – Part (A).

3.2.2 Plate with Two Opposite Edges Simply Supported and the Other Two Clamped (SSCC)

For this plate case, the plate has been chosen to be simply supported at the boundaries, $x = -a/2$ & $x = a/2$ and clamped at the boundaries $y = -b/2$ & $y = b/2$. These boundaries result a total of 8 boundary conditions, 2 at each boundary, which are:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (w_y)_{y=-b/2} = (\frac{\partial w}{\partial y})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (w_y)_{y=-b/2} = (\frac{\partial w}{\partial y})_{y=b/2} = 0 \tag{3.3}$$

The SSCC plate case was solved in this study one time with Galerkin method and one time with Ritz method using all the boundary conditions to be satisfied in both solutions.

In both methods, the starting assumed function of deflection was used to be:

$$w = \sum_{i=0}^{n+1} \sum_{j=0}^{n} C_{i,j}(a^2 - 4x^2)(b^2 - 4y^2)^2 x^{2i} y^{2j} \tag{3.4}$$

The function has been chosen to satisfy all the boundary conditions and gives symmetric solution in x and y directions which reduces the time consumed by the software.

Both methods were tested at values of n starting from $n = 1$ to $n = 4$, and found that at $n = 4$ both methods give brilliant results compared to the results of Timoshenko [1]. The comparison of the results at critical points are given in Table 5 & Table 6.

Going over these tables (Table 5 & Table 6) shows clearly the accuracy of the used methods to derive plate solutions, since the results perfectly matches with the results derived by Timoshenko.

Table 5 Comparison of results for uniformly loaded rectangular SSCC plate (Galerkin – G & Timoshenko - T) for ($\nu = 0.3$), $n = 4$

$n = 4$					
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$M_{y(0,\frac{b}{2})}$ $* qa^2$
0.5	G	0.00016316	0.00353458	0.0104987	-0.0210497
	T	0.00016	0.00355	0.0105	-0.02105
	% Diff.	-0.41%	0.43%	0.01%	0.00%
0.714	G	0.00062828	0.00970859	0.020366	-0.0411959
	T	0.00062	0.00979592	0.02035714	-0.0413265
	% Diff.	-0.57%	0.89%	-0.04%	0.32%
0.833	G	0.00107272	0.0149907	0.0259434	-0.0535319
	T	0.00108	0.01493056	0.02604167	-0.0535417
	% Diff.	0.25%	-0.40%	0.38%	0.02%
1	G	0.0019172	0.0244078	0.0332664	-0.0698348
	T	0.00192	0.0244	0.0332	-0.0697
	% Diff.	0.15%	-0.03%	-0.20%	-0.19%
1.2	G	0.00319411	0.0376245	0.0400007	-0.0875494
	T	0.00319	0.0376	0.04	-0.0868
	% Diff.	-0.13%	-0.07%	0.00%	-0.86%
1.5	G	0.00532631	0.0584244	0.0459207	-0.104817
	T	0.00531	0.0585	0.046	-0.1049
	% Diff.	-0.31%	0.13%	0.17%	0.08%
1.8	G	0.00731798	0.0769733	0.0477205	-0.113662
	T	0.00732	0.0768	0.0477	-0.1152
	% Diff.	0.03%	-0.23%	-0.04%	1.34%
2	G	0.00844502	0.086873	0.0473678	-0.119128
	T	0.00844	0.0869	0.0474	-0.1191
	% Diff.	-0.06%	0.03%	0.07%	-0.02%
3	G	0.0116821	0.114578	0.0422209	-0.124846
	T	0.01168	0.1144	0.0419	-0.1246
	% Diff.	-0.02%	-0.16%	-0.77%	-0.20%

Table 6 Comparison of results for uniformly loaded rectangular SSCC plate (Ritz – R & Timoshenko - T) for ($\nu = 0.3$), $n = 4$

$n = 4$					
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{y(0,\frac{b}{2})} * qa^2$
0.5	R	0.000163165	0.00353677	0.0104988	-0.0210161
	T	0.00016	0.00355	0.0105	-0.02105
	% Diff.	-0.41%	0.37%	0.01%	0.16%
0.714	R	0.00062813	0.00964798	0.020313	-0.041269
	T	0.00062	0.009795918	0.020357143	-0.041326531
	% Diff.	-0.54%	1.51%	0.22%	0.14%
0.833	R	0.0010726	0.0149216	0.025991	-0.0534352
	T	0.00108	0.014930556	0.026041667	-0.053541667
	% Diff.	0.26%	0.06%	0.19%	0.20%
1	R	0.00191702	0.0243438	0.0332124	-0.0697941
	T	0.00192	0.0244	0.0332	-0.0697
	% Diff.	0.16%	0.23%	-0.04%	-0.14%
1.2	R	0.00319408	0.0375544	0.0400274	-0.0871184
	T	0.00319	0.03760	0.04000	-0.08680
	% Diff.	-0.13%	0.12%	-0.07%	-0.37%
1.5	R	0.00532644	0.058481	0.0459423	-0.104843
	T	0.00531	0.0585	0.046	-0.1049
	% Diff.	-0.31%	0.03%	0.13%	0.05%
1.8	R	0.00731688	0.0767998	0.0476236	-0.115014
	T	0.00732	0.0768	0.0477	-0.1152
	% Diff.	0.04%	0.00%	0.16%	0.16%
2	R	0.00844499	0.0868692	0.0473617	-0.119171
	T	0.00844	0.0869	0.0474	-0.1191
	% Diff.	-0.06%	0.04%	0.08%	-0.06%
3	R	0.0116812	0.114356	0.042118	-0.124772
	T	0.01168	0.1144	0.0419	-0.1246
	% Diff.	-0.01%	0.04%	-0.52%	-0.14%

3.2.3 Plate with Two Opposite Edges Simply Supported and the Other Two Free (SSFF)

SSFF plate case is more complicated than the previous two cases since it includes two free edges which results in difficult boundary conditions. In this study, the plate has been chosen to be simply supported at the boundaries, $x = -a/2$ & $x = a/2$ and free at the boundaries $y = -b/2$ & $y = b/2$. These boundaries result the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \ \& \ (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \ \& \ (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0 \quad (3.5)$$

As a result of the complexity of this plate case, it has been solved by Timoshenko [1] for only three values of b/a. In this study, it has been solved one time with Galerkin method and one time with Ritz method using all the boundary conditions to be satisfied in both solutions.

In both methods, the starting assumed function of deflection was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j}(a^2 - 4x^2)x^{2i}y^{2j} \quad (3.6)$$

This function surely satisfies the simply supported boundary conditions, while the remaining boundary conditions are satisfied by mathematica software. Moreover, the given equation provides symmetric solution in x and y directions which reduces the time consumed by the software.

Both methods were tested at values of n starting from $n = 1$ to $n = 4$, and found that at $n = 4$ both methods give great results compared to the results of Timoshenko [1]. The comparison of the results at critical points are given in Table 7.

Table 7 Comparison of results for uniformly loaded rectangular SSFF plate (Galerkin – G, Ritz – R & Timoshenko - T) for ($\nu = 0.3$), $n = 4$

$n = 4$							
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$
0.5	G	0.0137146	0.124068	0.0116197	0.0146427	0.127338
	T	0.01377	0.1235	0.0102	0.01443	0.1259
	% Diff.	0.40%	-0.46%	-13.92%	-1.47%	-1.14%
1	G	0.013094	0.122635	0.0271145	0.0150077	0.130535
	T	0.01309	0.1225	0.0271	0.01509	0.1318
	% Diff.	-0.03%	-0.11%	-0.05%	0.55%	0.96%
2	G	0.0128997	0.124647	0.0371407	0.0152249	0.134284
	T	0.01289	0.1235	0.0364	0.01521	0.1329
	% Diff.	-0.08%	-0.93%	-2.03%	-0.10%	-1.04%
n=4						
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$
0.5	R	0.0137158	0.124222	0.0126599	0.0146381	0.127138
	T	0.01377	0.1235	0.0102	0.01443	0.1259
	% Diff.	0.39%	-0.58%	-24.12%	-1.44%	-0.98%
1	R	0.0130932	0.122469	0.0271694	0.0150097	0.130686
	T	0.01309	0.1225	0.0271	0.01509	0.1318
	% Diff.	-0.02%	0.03%	-0.26%	0.53%	0.85%
2	R	0.0128901	0.123858	0.0365491	0.0152003	0.132392
	T	0.01289	0.1235	0.0364	0.01521	0.1329
	% Diff.	0.00%	-0.29%	-0.41%	0.06%	0.38%

Both methods show a good match with the results of Timoshenko except for the value of $M_{y(0,0)}$ for $b/a = 0.5$. However, analysis of this specific plate case with the finite element software, COMSOL, shows that the value of $M_{y(0,0)}$ for $b/a = 0.5$ should be equal to 0.0118528 which is far away from the result given by Timoshenko and very close to the results derived by this study, which means that something done wrongly in Timoshenko's solution resulted in a big error at this specific value and should be modified. After discovering this error in Timoshenko's solution, a search has been done to see if there is any available article that goes over this error and states the reasons behind it. Fortunately, an article with the name "An Error in Timoshenko's 'Theory of Plates and Shells'" done by Angus Ramsay and Edward Maunder [20] has been found. This article discusses this error in details and states that the reason behind this error is probably typographical in the table and not in the solution equation. Hence, it needs modification in the original book

3.2.4 Plate with Three Edges Simply Supported and the Fourth Clamped (SSSC)

Starting from this plate case, we move from symmetric plates in x and y to symmetric plates in one direction only. The boundaries of this case have been chosen to be simply supported at the boundaries, $x = -a/2$, $x = a/2$ & $y = -b/2$ and clamped at $y = b/2$. Therefore, the boundary conditions are:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (w_y)_{y=-b/2} = (\frac{\partial w}{\partial y})_{y=b/2} = 0 \tag{3.7}$$

As in the previous cases, this case was solved in this study one time with Galerkin method and one time with Ritz method using all the boundary conditions to be satisfied.

In both methods, the starting assumed function of deflection was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{2n} C_{i,j} (a^2 - 4x^2) x^{2i} y^j \quad (3.8)$$

The chosen function satisfies only the simply supported boundary conditions at $x = a/2$ and $x = -a/2$ while the remaining BCs are satisfied using the help of Mathematica software. It is clear from the equation that it gives symmetric solution about the y axis only since the boundaries are symmetric about y axis.

Both methods were tested at values of n starting from $n = 1$ to $n = 4$, and found that the results at $n = 4$ of both methods perfectly match with the results of Timoshenko [1]. The comparison of the results at critical points are given in Table 8.

Table 8 Comparison of results for uniformly loaded rectangular SSSC plate (Galerkin – G, Ritz – R & Timoshenko - T) for ($\nu = 0.3$), $n = 4$

$n = 4$					
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{y(0,\frac{b}{2})}$ $* qa^2$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$
2	G	0.00927796	-0.123904	0.0946001	0.0473368
	T	0.00930	-0.122	0.094	0.047
	% Diff.	0.24%	-1.56%	-0.64%	-0.72%
1.5	G	0.00644465	-0.112312	0.0689692	0.0477013
	T	0.00640	-0.112	0.069	0.048
	% Diff.	-0.70%	-0.28%	0.04%	0.62%
1.2	G	0.00426444	-0.0986296	0.048643	0.0444778
	T	0.00430	-0.098	0.049	0.044
	% Diff.	0.83%	-0.64%	0.73%	-1.09%
1	G	0.0027864	-0.0839322	0.0340894	0.0393328
	T	0.0028	-0.084	0.034	0.039
	% Diff.	0.49%	0.08%	-0.26%	-0.85%
0.909	G	0.00216346	-0.0758203	0.0273459	0.0358038
	T	0.00218564	-0.0760331	0.02727273	0.03553719
	% Diff.	1.01%	0.28%	-0.27%	-0.75%
0.769	G	0.00132795	-0.0609942	0.0183279	0.0294906
	T	0.00133049	-0.0609467	0.0183432	0.0295858
	% Diff.	0.19%	-0.08%	0.08%	0.32%
0.667	G	0.00083943	-0.0494298	0.0126653	0.024175
	T	0.00082963	-0.0493333	0.01244444	0.024
	% Diff.	-1.18%	-0.20%	-1.77%	-0.73%

$n = 4$						
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{y(0,\frac{b}{2})}$ $* qa^2$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$
2	R	0.00927073	-0.121526	0.0941798	0.0468989
	T	0.00930	-0.122	0.094	0.047
	% Diff.	0.31%	0.39%	-0.19%	0.22%
1.5	R	0.00644467	-0.112279	0.0689837	0.0477019
	T	0.00640	-0.112	0.069	0.048
	% Diff.	-0.70%	-0.25%	0.02%	0.62%
1.2	R	0.00426419	-0.0982424	0.0486201	0.0444367
	T	0.00430	-0.098	0.049	0.044
	% Diff.	0.83%	-0.25%	0.78%	-0.99%
1	R	0.00278529	-0.0842081	0.0338359	0.0391711
	T	0.00280	-0.084	0.034	0.039
	% Diff.	0.53%	-0.25%	0.48%	-0.44%
0.909	R	0.00216321	-0.0755789	0.0273402	0.0357417
	T	0.00219	-0.0760331	0.02727273	0.03553719
	% Diff.	1.03%	0.60%	-0.25%	-0.58%
0.769	R	0.00132787	-0.0609561	0.0183074	0.0294691
	T	0.00133	-0.0609467	0.0183432	0.0295858
	% Diff.	0.20%	-0.02%	0.20%	0.39%
0.667	R	0.00083919	-0.0492706	0.0126148	0.024098
	T	0.00083	-0.0493333	0.01244444	0.024
	% Diff.	-1.15%	0.13%	-1.37%	-0.41%

Table 8 is a good proof of the accuracy of both methods so far.

This plate case is the first studied case having the plate symmetric in one direction only (in our case it is symmetric about the y axis), and it has been chosen in this thesis to show the simplicity and the practicality of dealing with the derived polynomial solutions for design optimization. In order to design any plate, it is needed to locate the maximum moments over the plate (which causes maximum stresses) and their magnitude. As an example, let's consider an SSSC plate with $b/a = 2$. The derived Galerkin polynomial solution for this ratio with $v = 0.3$ and $n = 4$ is given in Appendix D - Part (B).

Since the solution is in the form of polynomial, then M_x & M_y solutions over the plate can be easily derived using equations (2.1) & (2.2) which were previously given in Section 2.2 and they are basically derivatives of w:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) \tag{2.1}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) \tag{2.2}$$

Since the studied plate is symmetric about the y axis and not symmetric about the x axis because of boundaries conditions distribution, then for sure the maximum moments are not at the center of the plate as in the previous cases. Starting with M_x, it is expected that its maximum value is somewhere along the line $x = 0$ between the center of the plate and the simply supported edge at $y = -1$. The plot of the deflection w and M_x along the line $x = 0$ can be easily plotted using Mathematica software by substituting $x = 0$ in the equations of the deflection w and the moment M_x as shown in Figure 5 & Figure 6.

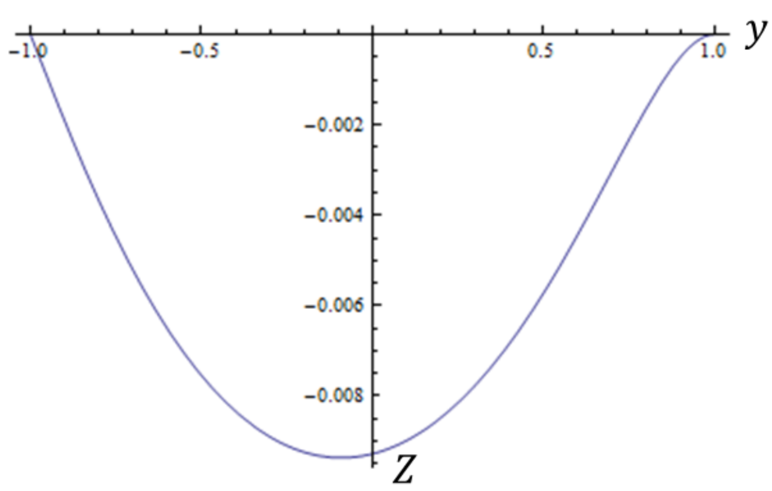

Figure 5 Plot of the Deflection w at the Centerline $x = 0$ for uniformly loaded SSSC Plate

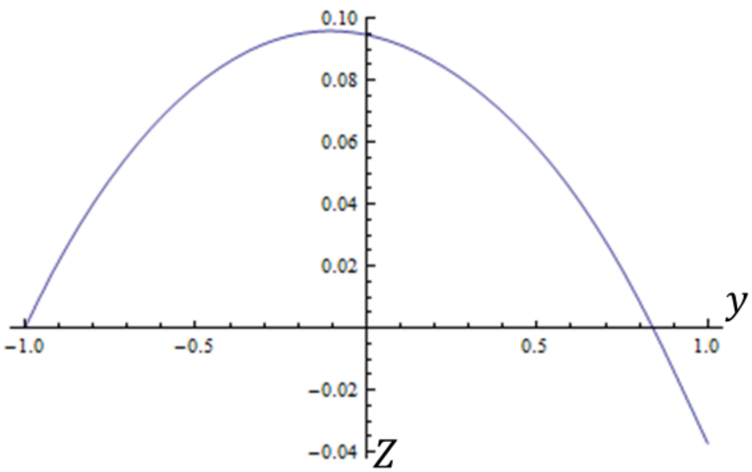

Figure 6 Plot of the Moment M_x at the Centerline $x = 0$ for uniformly loaded SSSC Plate

The exact location of the maximum M_x can be easily found by taking the derivative of the equation of M_x with respect to y and setting it to be equal to 0 ($\left.\frac{dM_x}{dy}\right|_{x=0} = 0$). Solving the previous equation results that the maximum M_x is at $y = -0.1061$ and substituting back in gives the maximum value of M_x to be $0.0958 * qa^2$.

The same procedure can be repeated for getting the maximum M_y. It is expected that the maximum value of it is somewhere along the line $x = 0$ either at the clamped edge at $y = 1$ or between the center of the plate and the simply supported edge at $y = -1$. The plot of the moment M_y along the line $x = 0$ can be easily plotted using Mathematica software by substituting $x = 0$ in the equation M_y as shown in Figure 7. It is clear from the figure that the maximum value of the moment M_y is at the center of the clamped edge. Substituting $x = 0$ and $y = 1$ in the equation of M_y gives the maximum value of M_y to be equal to $-0.1239 * qa^2$.

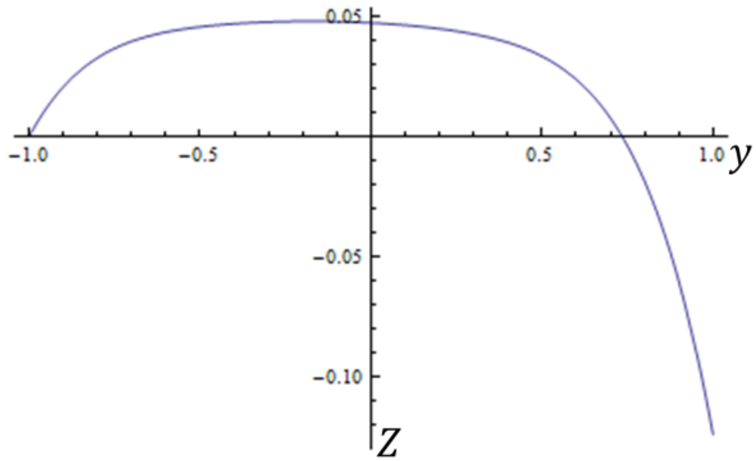

Figure 7 Plot of the Moment M_y at the Centerline $x = 0$ for uniformly loaded SSSC Plate

To show more clearly the practicality of the derived solution over the previous available solution, lets compare the way of finding the locations and magnitudes of maximum moments with that of Timoshenko's solution. For uniformly loaded SSSC plate, Timoshenko's solution comes in the form:

$$w = \sum_{m=1}^{\infty}(w_1 + w_2) \tag{3.9}$$

where:

$$w1 = \frac{4qa^4}{\pi^5 \mathrm{DD}}\frac{1}{m^5}\mathrm{Sin}[\frac{m\pi(x+a/2)}{a}]$$

$$w2 = \frac{qa^4}{\mathrm{DD}}\mathrm{Sin}[\frac{m\pi(x+a/2)}{a}](\mathrm{AmCosh}[\frac{m\pi y}{a}] + \mathrm{Bm}\frac{m\pi y}{a}\mathrm{Sinh}[\frac{m\pi y}{a}] + \mathrm{CmSinh}[\frac{m\pi y}{a}] + \mathrm{Dm}\frac{m\pi y}{a}\mathrm{Cosh}[\frac{m\pi y}{a}])$$

$$Am = \frac{(-11bm\pi\mathrm{Cosh}[\frac{bm\pi}{2}] + 3bm\pi\mathrm{Cosh}[\frac{3bm\pi}{2}] + 2(-b^2m^2\pi^2 + 8\mathrm{Cosh}[bm\pi])\mathrm{Sinh}[\frac{bm\pi}{2}])}{(m^5\pi^5(2bm\pi - \mathrm{Sinh}[2bm\pi]))}$$

$$Bm = \frac{4(1 - 3\mathrm{Cosh}[bm\pi] + bm\pi\mathrm{Coth}[\frac{bm\pi}{2}])\mathrm{Sinh}[\frac{bm\pi}{2}]}{m^5\pi^5(2bm\pi - \mathrm{Sinh}[2bm\pi])}$$

$$Cm = -\frac{2b\mathrm{Cosh}[\frac{bm\pi}{2}](-bm\pi + \mathrm{Sinh}[bm\pi])}{m^4\pi^4(-2bm\pi + \mathrm{Sinh}[2bm\pi])}$$

$$Dm = \frac{4\mathrm{Sinh}[\frac{bm\pi}{2}](-bm\pi + \mathrm{Sinh}[bm\pi])}{m^5\pi^5(-2bm\pi + \mathrm{Sinh}[2bm\pi])}$$

In order to get the locations and magnitudes of maximum moments using Timoshenko's solution, the previously showed procedure should be followed which requires finding the derivatives of w and this is not an easy job since the solution of w is in the form of trigonometric and hyperbolic series solution. Doing that is a time and effort consuming work and at the end will give very similar results.

3.2.5 Plate with Three Edges Simply Supported and the Fourth Free (SSSF)

This plate case is another case that is symmetric in one direction only. The boundaries of this case have been chosen to be simply supported at the boundaries, $x = -a/2, x = a/2$ & $y = -b/2$ and free at $y = b/2$. These boundaries provide the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \; \& \; (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0 \quad (3.10)$$

In this study, this case has been solved with both the Galerkin method and the Ritz method using all the boundary conditions to be satisfied.

The starting assumed function of deflection for both methods was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{2n+2} C_{i,j}(a^2 - 4x^2)x^{2i}y^j \quad (3.11)$$

This function satisfies only the simply supported boundary conditions at $x = a/2$ and $x = -a/2$ while the remaining BCs are satisfied using the help of Mathematica software. The equation provides symmetric solution about the y axis only since the boundaries are symmetric about y axis.

Both methods were tested at values of n starting from $n = 1$ to $n = 3$, and found that the results at $n = 3$ of both methods match the results of Timoshenko [1]. The comparison of the results at critical points is given in Table 9.

Table 9 Comparison of results for uniformly loaded rectangular SSSF plate (Galerkin – G, Ritz – R & Timoshenko - T) for ($\nu = 0.3$), $n = 3$

$n = 3$					
b/a	Method	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	G	0.00709023	0.0595781	0.0382286	0.0222027
	T	0.00710	0.06	0.039	0.022
	% Diff.	0.14%	0.70%	1.98%	-0.92%
0.77	G	0.0109188	0.0940984	0.0642429	0.0341125
	T	0.01092	0.094	0.064	0.034
	% Diff.	0.01%	-0.10%	-0.38%	-0.33%
1	G	0.0128489	0.111159	0.0799879	0.0390591
	T	0.01286	0.112	0.08	0.039
	% Diff.	0.09%	0.75%	0.02%	-0.15%
1.2	G	0.0138354	0.120313	0.0907457	0.0414648
	T	0.01384	0.121	0.09	0.041
	% Diff.	0.03%	0.57%	-0.83%	-1.13%
1.5	G	0.0146102	0.127158	0.10167	0.0422758
	T	0.01462	0.128	0.101	0.042
	% Diff.	0.07%	0.66%	-0.66%	-0.66%
2	G	0.0150275	0.128051	0.110545	0.0402965
	T	0.01507	0.132	0.113	0.041
	% Diff.	0.28%	2.99%	2.17%	1.72%
3	G	0.0152501	0.135275	0.122	0.0393961
	T	0.0152	0.133	0.122	0.039
	% Diff.	-0.33%	-1.71%	0.00%	-1.02%

$n = 3$					
b/a	Method	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	R	0.00708967	0.0595359	0.0380463	0.0220639
	T	0.00710	0.06	0.039	0.022
	% Diff.	0.15%	0.77%	2.45%	-0.29%
0.77	R	0.0109097	0.0933962	0.0626074	0.0343604
	T	0.01092	0.094	0.064	0.034
	% Diff.	0.09%	0.64%	2.18%	-1.06%
1	R	0.012846	0.110935	0.0799825	0.0389479
	T	0.01286	0.112	0.08	0.039
	% Diff.	0.11%	0.95%	0.02%	0.13%
1.2	R	0.0138342	0.120226	0.0903881	0.0413041
	T	0.01384	0.12100	0.09000	0.04100
	% Diff.	0.04%	0.64%	-0.43%	-0.74%
1.5	R	0.0146139	0.127445	0.101522	0.0423251
	T	0.01462	0.128	0.101	0.042
	% Diff.	0.04%	0.43%	-0.52%	-0.77%
2	R	0.0150701	0.131274	0.113984	0.0420905
	T	0.01507	0.132	0.113	0.041
	% Diff.	0.00%	0.55%	-0.87%	-2.66%
3	R	0.0152394	0.134461	0.122684	0.0396489
	T	0.01520	0.133	0.122	0.039
	% Diff.	-0.26%	-1.10%	-0.56%	-1.66%

Table 9 shows great results even though the number of terms (n) or the power of x and y is less than the terms used in the previous cases.

3.2.6 Plate with Two Opposite Edges Simply Supported, One Clamped and One Free (SSCF)

This plate is the last case in this chapter and it is another case that is symmetric in one direction only. The boundaries of this case have been chosen to be simply supported at the boundaries, $x = -a/2$ & $x = a/2$, clamped at $y = -b/2$ and free at $y = b/2$. These boundaries result the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (w_y)_{y=-b/2} = (\frac{\partial w}{\partial y})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \; \& \; (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0 \quad (3.12)$$

Galerkin and Ritz methods were used to solve this case using all the boundary conditions to be satisfied.

The following function of w has been chosen as the starting assumed function of deflection for Galerkin methods:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{8} C_{i,j}(a^2 - 4x^2)x^{2i}y^j \quad (3.13)$$

For the Ritz method, the same equation were used but with more terms in y:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{10} C_{i,j}(a^2 - 4x^2)x^{2i}y^j \quad (3.14)$$

These functions satisfy directly the simply supported edges boundary conditions while the remaining BCs are satisfied using the help of Mathematica software. The equations provide symmetric solution about the y axis only since the boundaries are symmetric about y axis.

Both methods were tested at values of n starting from $n = 1$ to $n = 4$, and found that the results at $n = 4$ of both methods match the results of Timoshenko [1]. The comparison of the results at critical points are given in Table 10.

Table 10 Comparison of results for uniformly loaded rectangular SSCF plate (Galerkin – G, Ritz – R & Timoshenko - T) for ($\nu = 0.3$), $n = 4$

$n = 4$				
b/a	**Method**	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{y(0,-\frac{b}{2})} * qa^2$
0.5	G	0.00363445	0.0286158	-0.080809
	T	0.00364	0.0293	-0.07975
	% Diff.	0.08%	2.34%	-1.33%
1	G	0.0112374	0.097012	-0.118974
	T	0.01130	0.0972	-0.119
	% Diff.	0.55%	0.19%	0.02%
1.5	G	0.0141523	0.123354	-0.125013
	T	0.01410	0.123	-0.124
	% Diff.	-0.37%	-0.29%	-0.82%
2	G	0.0149848	0.132516	-0.124922
	T	0.01500	0.131	-0.125
	% Diff.	0.10%	-1.16%	0.06%
3	G	0.0152829	0.135675	-0.121153
	T	0.0152	0.133	-0.125
	% Diff.	-0.55%	-2.01%	3.08%
n=4				
b/a	**Method**	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{y(0,-\frac{b}{2})} * qa^2$
0.5	R	0.00363873	0.0291722	-0.0785383
	T	0.00364	0.0293	-0.07975
	% Diff.	-0.03%	0.44%	1.52%
1	R	0.0112358	0.0969276	-0.11883
	T	0.01130	0.0972	-0.119
	% Diff.	0.57%	0.28%	0.14%
1.5	R	0.0141504	0.123275	-0.124856
	T	0.01410	0.123	-0.124
	% Diff.	-0.36%	-0.22%	-0.69%
2	R	0.014952	0.130431	-0.125977
	T	0.01500	0.131	-0.125
	% Diff.	0.32%	0.43%	-0.78%
3	R	0.0152327	0.134701	-0.127061
	T	0.01520	0.133	-0.125
	% Diff.	-0.22%	-1.28%	-1.65%

Table 10 shows a negligible difference in results between the studied methods and the Timoshenko's solution and proof the applicability of the used methods.

3.3 Closure

It is noted that for the above six cases, Timoshenko in his book [1] has provided tables with the values of the maximum deflections and moments at certain points and for some cases he even did not represent the exact analytical solutions. Comparison of Timoshenko tables with the results of the two approaches developed here showed excellent matching which indicates the validity, accuracy and applicability of the developed approaches. Moreover, the used approaches provide us with exact deflection equations for each of the previously analyzed plate cases and from them, exact expressions of bending moments and stresses can be derived by applying the basic relations discussed in section 2.2. Plate cases discussed in this chapter have existing solutions in Timoshenko book and that made the testing and evaluation of the derived solutions an easy job. However, this is not the situation for some of the remaining plate cases with other types of boundary conditions, which increases the analysis challenge in the following chapters of this research.

CHAPTER 4

UNIFORMLY LOADED RECTANGULAR PLATES WITH TWO OPPOSITE EDGES CLAMPED

4.1 Introduction

After analyzing all the cases of uniformly loaded rectangular plates having two opposite boundaries simply supported, the analysis moves to the cases having two opposite boundaries clamped. This includes 5 types of cases that are well analyzed in this chapter. Figure 8 shows the configuration and the coordinate system for the plates being analyzed in this chapter. The plate is clamped at $x = -a/2$ and $x = a/2$, while the other boundaries at $y = -b/2$ and $y = b/2$ can be either simply supported, clamped or free.

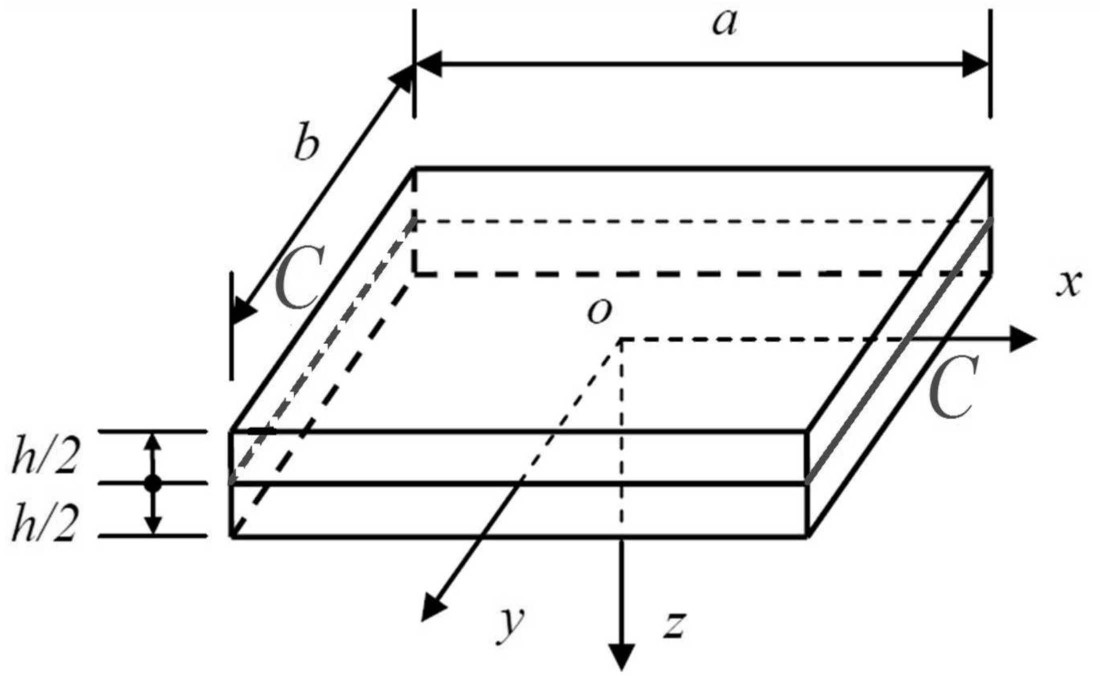

Figure 8 Configuration of Plates with Two Opposite Edges Clamped

4.2 Plate Bending Solutions and Comparison of Results

The methodology and the codes used to derive solutions for uniformly loaded rectangular plates with two opposite boundaries clamped were previously shown and explained in sections 2.4 and 2.5. It includes the same procedure used in chapter 0 with only few differences related to the boundary conditions and the assumed starting deflection function (w). This section presents the bending solutions for five plates cases with two opposite sides simply supported and compare these results with the solution derived by Timoshenko in his book "Theory of Plates and Shells" [1] or the FEM solution derived using COMSOL Multiphysics. Two of these cases were not discussed in Timoshenko book but this research succeeded to get out with an accurate solution for these missing cases as they have been proved by comparison with the FEM solution derived by

COMSOL Multiphysics. As a part of the methodology used in the study, the values of D (the flexural rigidity of the plate), q (uniform load), a (width of the plate) are always used to be 1 since they are used as scaling parameters. As a result of that, the derived solutions for the deflection, moments and shears of the plates are in the form of functions that are polynomials of x & y multiplied by the scaling parameters$a$, q and D raised to some power. The value of b (length of the plate) is set to be equal to the ratio of b/a and in most of the cases in this chapter, Poisson's Ratio (ν) is set to have the value 0.3 (unless stated to different with some reasons).

4.2.1 Plate Clamped from All the Four Edges (CCCC)

Uniformly loaded fully clamped plate is the case chosen to be solved first in this chapter because it has opposite clamped sides in both directions. Therefore, the plate is clamped at the boundaries, $x = -a/2$, $x = a/2$, $y = -b/2$ and $y = b/2$. These four boundaries give a total of 8 boundary conditions, 2 at each boundary as discussed in section 2.3, which are:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (w_x)_{x=a/2} = (\frac{\partial w}{\partial x})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (w_y)_{y=-b/2} = (\frac{\partial w}{\partial y})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (w_y)_{y=b/2} = (\frac{\partial w}{\partial y})_{y=b/2} = 0 \qquad (4.1)$$

It is obvious from the previous BCs that all of them are essential BCs and therefore, to find a solution for the plate using the Galerkin method or the Ritz method, all the boundary conditions must be satisfied. This plate case was solved in this study one time with Galerkin method and one time with Ritz method.

In both methods, the starting assumed function of deflection was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} (a^2 - 4x^2)^2 (b^2 - 4y^2)^2 x^{2i} y^{2j} \tag{4.2}$$

This function has been specifically chosen because it surely satisfies all the boundary conditions and gives symmetric solution in x and y directions which reduces the time consumed by the software.

Both methods were tested at values of n starting from $n = 1$ to $n = 4$, and found that the results at $n = 4$ of both methods match the results of Timoshenko [1]. The comparison of the results at critical points are given in Table 11.

The matching of results between the study methods and the Timoshenko's solution in Table 11 shows the accuracy and the applicability of the study methods.

Table 11 Comparison of results for uniformly loaded rectangular CCCC plate (Galerkin – G, Ritz – R & Timoshenko - T) for ($\nu = 0.3$), $n = 4$

$n = 4$						
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(\frac{a}{2},0)}$ $* qa^2$	$M_{y(0,\frac{b}{2})}$ $* qa^2$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$
1	G	0.00126532	-0.051299	-0.051299	0.0229031	0.0229031
	T	0.00126	-0.0513	-0.0513	0.0231	0.0231
	% Diff.	-0.42%	0.00%	0.00%	0.85%	0.85%
1.2	G	0.00172487	-0.0638377	-0.0553833	0.029969	0.0228382
	T	0.00172	-0.0639	-0.0554	0.0299	0.0228
	% Diff.	-0.28%	0.10%	0.03%	-0.23%	-0.17%
1.5	G	0.00219652	-0.0755648	-0.0570133	0.0367683	0.0202658
	T	0.00220	-0.0757	-0.057	0.0368	0.0203
	% Diff.	0.16%	0.18%	-0.02%	0.09%	0.17%
1.7	G	0.00238202	-0.0797275	-0.0570942	0.0392662	0.0182673
	T	0.00238	-0.0799	-0.0571	0.0392	0.0182
	% Diff.	-0.08%	0.22%	0.01%	-0.17%	-0.37%
2	G	0.00253295	-0.0827339	-0.0569873	0.0411513	0.0158068
	T	0.00254	-0.0829	-0.0571	0.0412	0.0158
	% Diff.	0.28%	0.20%	0.20%	0.12%	-0.04%
$n = 4$						
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(\frac{a}{2},0)}$ $* qa^2$	$M_{y(0,\frac{b}{2})}$ $* qa^2$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$
1	R	0.00126532	-0.0512994	-0.0512994	0.0229031	0.0229031
	T	0.00126	-0.0513	-0.0513	0.0231	0.0231
	% Diff.	-0.42%	0.00%	0.00%	0.85%	0.85%
1.2	R	0.00172487	-0.0638374	-0.0553827	0.0299689	0.0228381
	T	0.00172	-0.0639	-0.0554	0.0299	0.0228
	% Diff.	-0.28%	0.10%	0.03%	-0.23%	-0.17%
1.5	R	0.00219652	-0.0755649	-0.0570138	0.0367684	0.0202659
	T	0.00220	-0.0757	-0.057	0.0368	0.0203
	% Diff.	0.16%	0.18%	-0.02%	0.09%	0.17%
1.7	R	0.00238202	-0.0797277	-0.0570942	0.0392662	0.0182673
	T	0.00238	-0.0799	-0.0571	0.0392	0.0182
	% Diff.	-0.08%	0.22%	0.01%	-0.17%	-0.37%
2	R	0.00253295	-0.0827341	-0.0569873	0.0411513	0.0158069
	T	0.00254	-0.0829	-0.0571	0.0412	0.0158
	% Diff.	0.28%	0.20%	0.20%	0.12%	-0.04%

4.2.2 Plate with Three Edges Clamped and the Fourth Simply Supported (CCCS)

For this plate case, the plate has been chosen to be clamped at the boundaries, $x = -a/2$, $x = a/2$ & $y = b/2$ and simply supported at the boundary $y = -b/2$. These boundaries result a total of 8 boundary conditions, 2 at each boundary, which are:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (w_x)_{x=a/2} = (\frac{\partial w}{\partial x})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (w_y)_{y=b/2} = (\frac{\partial w}{\partial y})_{y=b/2} = 0 \tag{4.3}$$

The CCCS plate case was solved in this study one time with Galerkin method and one time with Ritz method using all the boundary conditions to be satisfied in both solutions.

In both methods, the starting assumed function of deflection was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{2n+3} C_{i,j} (a^2 - 4x^2)^2 x^{2i} y^j \tag{4.4}$$

The chosen function satisfies only the clamped boundary conditions at $x = a/2$ and $x = -a/2$ while the remaining BCs are satisfied using the help of Mathematica software. It is clear from the equation that it gives symmetric solution about the y axis only since the boundaries are symmetric about y axis.

Both methods were tested at values of n starting from $n = 1$ to $n = 5$, and found that at $n = 5$ both methods give brilliant results compared to the results of Timoshenko [1]. The comparison of the results at critical points are given in Table 12.

Table 12 Comparison of results for uniformly loaded rectangular CCCS plate (Galerkin – G, Ritz – R & Timoshenko - T) for ($\nu = 0.3$), $n = 5$

$n = 5$				
b/a	**Method**	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(\frac{a}{2},0)} * qa^2$	$M_{y(0,\frac{b}{2})} * qa^2$
0.5	G	0.00028057	-0.0195992	-0.0286496
	T	0.00028	-0.01965	-0.0287
	% Diff.	0.02%	0.26%	0.18%
0.75	G	0.00090603	-0.0406906	-0.0471178
	T	0.00090	-0.0410625	-0.0471375
	% Diff.	-0.12%	0.91%	0.04%
1	G	0.00157045	-0.0599467	-0.0550749
	T	0.00157	-0.0601	-0.0551
	% Diff.	-0.03%	0.26%	0.05%
1.33	G	0.00216217	-0.0748803	-0.0570272
	T	0.00215	-0.075	-0.0571
	% Diff.	-0.57%	0.16%	0.13%
2	G	0.00257192	-0.0833942	-0.0570364
	T	0.00257	-0.0837	-0.0571
	% Diff.	-0.07%	0.37%	0.11%
$n = 5$				
b/a	**Method**	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(\frac{a}{2},0)} * qa^2$	$M_{y(0,\frac{b}{2})} * qa^2$
0.5	R	0.00028056	-0.0196021	-0.0286274
	T	0.00028	-0.01965	-0.0287
	% Diff.	0.02%	0.24%	0.25%
0.75	R	0.00090616	-0.0410044	-0.0470321
	T	0.00090	-0.0410625	-0.0471375
	% Diff.	-0.14%	0.14%	0.22%
1	R	0.00157052	-0.0599569	-0.0549602
	T	0.00157	-0.0601	-0.0551
	% Diff.	-0.03%	0.24%	0.25%
1.33	R	0.00216205	-0.0749508	-0.0571289
	T	0.00215	-0.075	-0.0571
	% Diff.	-0.56%	0.07%	-0.05%
2	R	0.00257185	-0.0836369	-0.0571109
	T	0.00257	-0.0837	-0.0571
	% Diff.	-0.07%	0.08%	-0.02%

Going over Table 12 gives an indication of the accuracy of the used methods to derive plate solutions even though the compared values are few.

4.2.3 Plate with Three Edges Clamped and the Fourth Free (CCCF)

This plate case is another case that is symmetric in one direction only. The boundaries of this case have been chosen to be clamped at the boundaries, $x = -a/2$, $x = a/2$ & $y = -b/2$ and free at $y = b/2$. These boundaries provide the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (w_x)_{x=a/2} = (\frac{\partial w}{\partial x})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (w_y)_{y=-b/2} = (\frac{\partial w}{\partial y})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \; \& \; (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0 \quad (4.5)$$

In this study, this case was not able to be solved with Mathematica by applying all the boundary conditions. This is due to the high constraints that these boundaries cause for the solution of this plate case, especially the BCs of the free edge. Therefore, in this study, this case has been solved only with the Ritz method using only the essential boundary conditions to be satisfied (deflection and slope at the clamped edges).

The starting assumed function of deflection for the solution was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{2n} C_{i,j} x^{2i} y^{j} \qquad (4.6)$$

This function does not satisfy any of the boundary conditions, and they are left for Mathematica software to satisfy them. The equation provides symmetric solution about the y axis only since the boundaries are symmetric about y axis. In this plate case, Timoshenko in his book [1] used a value of Poisson's Ratio $\nu = \frac{1}{6}$, that's why the same value of ν has been used in the solution to make the results comparable.

The solution has been tested at values of n starting from $n = 1$ to $n = 4$, and found that the results even when n increases does not accurately match with the results of Timoshenko [1] as it is clear from the comparison given in Table 13 (values shaded with red color has % difference larger than 10%).

Table 13 shows a huge difference between the results derived in this study and the solution derived by Timoshenko. The minimum difference that in appears in the table is about 3%, which leaves a question mark on the used method and its accuracy. As a second check, the derived results were compared with the solution derived the FEM software COMSOL Multiphysics as shown in Table 14.

Table 13 Comparison of results for uniformly loaded rectangular CCCF plate (Ritz – R & Timoshenko - T) for $(\nu = 1/6)$, $n = 4$

$n = 4$							
b/a	Method	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(\frac{a}{2},\frac{b}{2})} * qa^2$
0.6	R	0.00222281	0.0321237	0.0010879	0.0166697	0.00733692	-0.0803648
	T	0.00271	0.03360	0.00129	0.01680	0.00740	-0.07450
	% Diff.	17.98%	4.39%	15.67%	0.78%	0.85%	-7.87%
0.7	R	0.0024871	0.0371115	0.00133259	0.0209503	0.0100728	-0.0842906
	T	0.00292	0.03710	0.00159	0.02120	0.00970	-0.07820
	% Diff.	14.83%	-0.03%	16.19%	1.18%	-3.84%	-7.79%
1	R	0.00276397	0.0424461	0.00189578	0.0306444	0.0135369	-0.083645
	T	0.00333	0.04440	0.00230	0.03170	0.01380	-0.08530
	% Diff.	17.00%	4.40%	17.57%	3.33%	1.91%	1.94%
1.25	R	0.00277684	0.043105	0.00220047	0.0354975	0.0129027	-0.0815981
	T	0.00345	0.04670	0.00269	0.03740	0.01420	-0.08670
	% Diff.	19.51%	7.70%	18.20%	5.09%	9.14%	5.88%
1.5	R	0.00276106	0.0431256	0.00239262	0.0385062	0.0113994	-0.0803459
	T	0.00335	0.04500	0.00290	0.04020	0.01180	-0.08420
	% Diff.	17.58%	4.17%	17.50%	4.21%	3.39%	4.58%

b/a	Method	$V_{x(\frac{a}{2},\frac{b}{2})} * qa$	$M_{x(\frac{a}{2},0)} * qa^2$	$V_{x(\frac{a}{2},0)} * qa$	$M_{y(0,-\frac{b}{2})} * qa^2$	$V_{y(,-\frac{b}{2})} * qa$
0.6	R	0.583133	-0.0412838	0.33014	-0.0540871	0.454828
	T	0.75000	-0.03650	0.29700	-0.05540	0.41600
	% Diff.	22.25%	-13.11%	-11.16%	2.37%	-9.33%
0.7	R	0.53175	-0.048992	0.374365	-0.0553823	0.456676
	T	0.71700	-0.04390	0.34600	-0.05450	0.41300
	% Diff.	25.84%	-11.60%	-8.20%	-1.62%	-10.58%
1	R	0.366299	-0.0670429	0.50093	-0.0563469	0.463912
	T	0.62800	-0.06140	0.43500	-0.05100	0.40100
	% Diff.	41.67%	-9.19%	-15.16%	-10.48%	-15.69%
1.25	R	0.317401	-0.0748629	0.512355	-0.0567243	0.46588
	T	0.57000	-0.07080	0.47500	-0.04700	0.38800
	% Diff.	44.32%	-5.74%	-7.86%	-20.69%	-20.07%
1.5	R	0.299645	-0.0794554	0.515907	-0.0568691	0.463666
	T	0.52700	-0.07550	0.49100	-0.04180	0.37300
	% Diff.	43.14%	-5.24%	-5.07%	-36.05%	-24.31%

Table 14 Comparison of results for uniformly loaded rectangular CCCF plate (Ritz – R & COMSOL) for $(\nu = 1/6)$, $n = 4$

$n = 4$						
b/a	Method	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.6	R	0.00222281	0.0321237	0.0010879	0.0166697	0.00733692
	COMSOL	0.00223443	0.03207707	0.00109056	0.01660431	0.00713079
	% Diff.	0.52%	-0.15%	0.24%	-0.39%	-2.89%
0.7	R	0.0024871	0.0371115	0.00133259	0.0209503	0.0100728
	COMSOL	0.00249829	0.03693503	0.00133529	0.02092381	0.00998687
	% Diff.	0.45%	-0.48%	0.20%	-0.13%	-0.86%
1	R	0.00276397	0.0424461	0.00189578	0.0306444	0.0135369
	COMSOL	0.00277446	0.04282776	0.00189762	0.03041642	0.01332916
	% Diff.	0.38%	0.89%	0.10%	-0.75%	-1.56%
1.25	R	0.00277684	0.043105	0.00220047	0.0354975	0.0129027
	COMSOL	0.00278654	0.04353761	0.00220305	0.03536445	0.01278143
	% Diff.	0.35%	0.99%	0.12%	-0.38%	-0.95%
1.5	R	0.00276106	0.0431256	0.00239262	0.0385062	0.0113994
	COMSOL	0.00277056	0.04344692	0.00239571	0.03842334	0.01133307
	% Diff.	0.34%	0.74%	0.13%	-0.22%	-0.59%
b/a	Method		$M_{x(\frac{a}{2},0)} * qa^2$	$V_{x(\frac{a}{2},0)} * qa$	$M_{y(0,-\frac{b}{2})} * qa^2$	$V_{y(,-\frac{b}{2})} * qa$
0.6	R		-0.0412838	0.33014	-0.0540871	0.454828
	T		-0.04139	0.32786	-0.05446	0.44730
	% Diff.		0.26%	-0.69%	0.69%	-1.68%
0.7	R		-0.048992	0.374365	-0.0553823	0.456676
	T		-0.04930	0.37790	-0.05574	0.45661
	% Diff.		0.61%	0.94%	0.64%	-0.01%
1	R		-0.0670429	0.50093	-0.0563469	0.463912
	T		-0.06650	0.46898	-0.05693	0.46521
	% Diff.		-0.82%	-6.81%	1.03%	0.28%
1.25	R		-0.0748629	0.512355	-0.0567243	0.46588
	T		-0.07501	0.50659	-0.05728	0.47173
	% Diff.		0.19%	-1.14%	0.97%	1.24%
1.5	R		-0.0794554	0.515907	-0.0568691	0.463666
	T		-0.07992	0.52612	-0.05744	0.47795
	% Diff.		0.58%	1.94%	0.99%	2.99%

When the derived solution compared with the FEM solution (which is most probably more accurate than Timoshenko's solution), they have shown a brilliant agreement which leaves no doubt that the error is in Timoshenko's solution and not in the method used in this research. The main reason behind the large errors in Timoshenko results is the crude approximations that Timoshenko used in his solution since he used the finite difference method to derive the solution.

4.2.4 Plate with Two Opposite Edges Clamped and the Other Two Free (CCFF)

CCFF plate case is a unique case among all the studied cases since it is the only 2 way symmetric plate case that does not have available solution in Timoshenko book [1]. This gives a high indication of the complexity of such a case and its difficulty of analysis. This difficulty is a result of the two opposite free edges. In this study, the plate has been chosen to be clamped at the boundaries, $x = -a/2$ & $x = a/2$ and free at the boundaries $y = -b/2$ & $y = b/2$. These boundaries result the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (w_x)_{x=a/2} = (\frac{\partial w}{\partial x})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \;\&\; (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \;\&\; (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0 \quad (4.7)$$

In this study, this case has been solved only one time with the Ritz method and by only using the essential boundary conditions to be satisfied. The restrictions provided by the BCs at the free edges did not allow the Galerkin method to derive a proper solution.

The starting assumed function of deflection was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} x^{2i} y^{2j} \tag{4.8}$$

This function surely provides symmetric solution for the plate, but the boundary conditions are satisfied by the help of Mathematica software.

The results were compared with the derived solution by COMSOL Multiphysics software for 5 b/a ratios and at values of n starting from $n = 1$ to $n = 5$ and found that at $n = 5$ both solutions give good matching. The comparison of the results at critical points are given in Table 15.

Table 15 Comparison of results for uniformly loaded rectangular CCFF plate (Ritz – R & COMSOL) for ($\nu = 0.3$), $n = 5$

$n = 5$							
b/a	Method	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})} * qa^2$	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(\frac{a}{2},0)} * qa^2$
0.5	R	0.00283159	0.0422164	0.00261764	0.0401188	0.00708848	-0.0848258
	COMSOL	0.00270728	0.04247299	0.00259802	0.04097486	0.00407301	-0.0835583
	% Diff.	-4.59%	0.60%	-0.76%	2.09%	-74.04%	-1.52%
0.75	R	0.0028815	0.0427433	0.00257351	0.0402817	0.00924145	-0.0817305
	COMSOL	0.00273473	0.04286075	0.00258127	0.04101633	0.005444	-0.0827151
	% Diff.	-5.37%	0.27%	0.30%	1.79%	-69.75%	1.19%
1	R	0.00290705	0.0431669	0.00255952	0.0407548	0.0110702	-0.0806956
	COMSOL	0.00274791	0.04306668	0.00257806	0.04111992	0.00626575	-0.0827486
	% Diff.	-5.79%	-0.23%	0.72%	0.89%	-76.68%	2.48%
1.25	R	0.00291597	0.0435572	0.00256279	0.0411467	0.0120469	-0.0811318
	COMSOL	0.00275252	0.04315949	0.00258262	0.04125384	0.00671501	-0.0831142
	% Diff.	-5.94%	-0.92%	0.77%	0.26%	-79.40%	2.39%
1.5	R	0.00291763	0.0437736	0.00257163	0.0411879	0.0123401	-0.0818279
	COMSOL	0.00275369	0.04319421	0.00258955	0.04138424	0.00692853	-0.0834692
	% Diff.	-5.95%	-1.34%	0.69%	0.47%	-78.11%	1.97%

Both solutions show a reasonable match at most of the compared results except for the maximum deflection, which has about 5% greater value in Ritz method solution, and in the magnitude of the bending moment around the y axis (M_y) at the center of the plate which has around 75% greater value in the solution of Ritz method. The difference error in the maximum deflection is acceptable and reasonable, while the difference in M_y is quite huge and needs further analysis to go over the reasons of it. One possible reason for that difference is the small magnitude of M_y compared to the magnitude of M_x over the whole plate. Very small values tend to have higher error because they are near zero. For instance, from FEM solution, at the center of the plate $M_y \approx 0.007$ while $M_x \approx 0.04$ and this is very big difference. This high difference makes finding exact values of M_y not very important and not critical in design. Another reason is due to the restriction provided at the free edges for the magnitude of M_y.

4.2.5 Plate with Two Opposite Edges Clamped, One Simply Supported and One Free (CCSF)

CCSF plate case is the second plate case discussed in this research and does not have available solution in Timoshenko's book [1]. As mentioned in the previous case, this may be considered as an indication of the difficulty of analysis for this case. In this case, the plate was given the following boundaries, clamped at the boundaries, $x = -a/2$ & $x = a/2$, simply supported at $y = -b/2$ and free at $y = b/2$. These boundaries provide the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (w_x)_{x=a/2} = (\frac{\partial w}{\partial x})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \ \& \ (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0 \quad (4.9)$$

These boundary conditions were not able to be solved in this thesis with the Galerkin method and therefore this case has been solved with the Ritz method only. It has been tried to develop Ritz method solution in two ways; one time by satisfying the BCs at the clamped and simply supported edges (6 BCs), and one time by satisfying only the essential BCs (BCs at clamped edges and w at the simply supported edge – 5 BCs). Both solutions gave almost similar numbers

The assumed starting function of deflection in both solutions was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{2n} C_{i,j} x^{2i} y^{j} \qquad (4.10)$$

This function does not satisfy any of the BCs by itself but it assure a symmetric solution for the plate around the y axis. The boundary conditions are satisfied by the help of Mathematica software.

Since there is not any available Timoshenko's solution for this case, the results were compared with the derived solution by COMSOL software for 5 b/a ratios, at values of n starting from $n = 1$ to $n = 5$ and found that at $n = 5$ both solutions give good matching. The comparison of the results of the Ritz method (only satisfying essential BCs) and COMSOL solution at critical points is shown in Table 16.

Table 16 Comparison of results for uniformly loaded rectangular CCSF plate (Ritz – R & COMSOL) for ($\nu = 0.3$), $n = 5$

$n = 5$					
b/a	Method	$w_{(0,\frac{b}{2})}$ $* \frac{qa^4}{D}$	$M_{x(0,\frac{b}{2})}$ $* qa^2$	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$
0.5	R	0.00264311	0.0365565	0.00150754	0.0247199
	COMSOL	0.00246132	0.0356699	0.00147492	0.02334075
	% Diff.	-7.39%	-2.49%	-2.21%	-5.91%
0.75	R	0.00295304	0.0436527	0.00195727	0.0326247
	COMSOL	0.00277526	0.04242144	0.00195816	0.03144649
	% Diff.	-6.41%	-2.90%	0.05%	-3.75%
1	R	0.00297301	0.0445827	0.00224665	0.0372075
	COMSOL	0.00280309	0.04367158	0.0022581	0.03626953
	% Diff.	-6.06%	-2.09%	0.51%	-2.59%
1.25	R	0.00294338	0.0430059	0.00242143	0.0391601
	COMSOL	0.00278019	0.04357202	0.00243681	0.03908742
	% Diff.	-5.87%	1.30%	0.63%	-0.19%
1.5	R	0.00292271	0.0429133	0.00252527	0.0413288
	COMSOL	0.0027633	0.04336392	0.00253587	0.04062477
	% Diff.	-5.77%	1.04%	0.42%	-1.73%
b/a	Method	$M_{y(0,0)}$ $* qa^2$	$M_{x(\frac{a}{2},\frac{b}{2})}$ $* qa^2$	$M_{x(\frac{a}{2},0)}$ $* qa^2$	$V_{x(\frac{a}{2},0)}$ $* qa$
0.5	R	0.0146616	-0.0768959	-0.0553112	-0.406731
	COMSOL	0.01331539	-0.0800756	-0.0550218	-0.4216088
	% Diff.	-10.11%	3.97%	-0.53%	3.53%
0.75	R	0.0178564	-0.0728787	-0.0679555	-0.46395
	COMSOL	0.01460579	-0.0745243	-0.0688533	-0.4855808
	% Diff.	-22.26%	2.21%	1.30%	4.45%
1	R	0.0179315	-0.0687016	-0.0764194	-0.547045
	COMSOL	0.01328554	-0.0696317	-0.0767306	-0.5111464
	% Diff.	-34.97%	1.34%	0.41%	-7.02%
1.25	R	0.0160079	-0.0669427	-0.0804946	-0.546756
	COMSOL	0.0113966	-0.0673915	-0.0810488	-0.5243837
	% Diff.	-40.46%	0.67%	0.68%	-4.27%
1.5	R	0.0151827	-0.0694329	-0.0821525	-0.51351
	COMSOL	0.0097645	-0.0666241	-0.083209	-0.5316561
	% Diff.	-55.49%	-4.22%	1.27%	3.41%

The comparison table in this case gives very much similar conclusions to the conclusions in the previous CCCF case. The table shows a reasonable match at most of the compared results even for shear and moment at the corner. Exceptions are for the maximum deflection, which has about 6% greater value in Ritz method solution, and in the magnitude of the bending moment around the y axis (M_y) at the center of the plate which is noticed to have higher % difference as b/a ratio increases. As mentioned in the CCCF case, for these two differences, the difference error in the maximum deflection is acceptable and reasonable, while the difference in M_y is quite huge and some possible reasons for that high difference were explained there.

4.3 Closure

In this chapter, the Ritz method used in this research was successful to derive solutions for all the plates cases with two opposite edges clamped. On the other hand, the Galerkin method was successful to derive solutions for only the cases without any free edge. The ability of applying the Ritz method without satisfying all the BCs (satisfying only the essential BCs) made it more flexible and gave this advantage for the Ritz method over the Galerkin method to exactly satisfy the general plate deflection equation. Comparison tables with Timoshenko [1] and FEM solutions were conducted to verify the validity and accuracy of the applied methods. Overall, the tables show excellent agreement except for some bending moment results that have big difference between the Ritz method solution and the FEM solution. The reasons behind that need more analysis and study. From the study it can be concluded that the Ritz method can successfully solve any plate problem so far, while the Galerkin method can solve plate cases that have few restrictions.

CHAPTER 5

UNIFORMLY LOADED RECTANGULAR PLATES WITH TWO OPPOSITE EDGES UNSYMMETRICAL

5.1 Introduction

In the previous two chapters, almost all the uniformly loaded rectangular plate cases with two opposite symmetric edges were analyzed. This chapter represents all the remaining cases with unsymmetrical opposite edges. These cases are divided into three main divisions, which are: plates with one edge simply supported and the opposite edge clamped, plates with one edge simply supported and the opposite edge free and plates with one edge clamped and the opposite edge free.

The methodology used to derive solutions in this chapter is exactly like the one used in the previous two chapters. The only few differences are in applying the boundary conditions and the assumed starting deflection function (w). This chapter presents the derived Galerkin and Ritz solutions for 9 plates cases and compare these results with the solution derived by Timoshenko in his book "Theory of Plates and Shells" [1] (if available) and the solution derived using FEM with the help of COMSOL Multiphysics. As a part of the methodology used in the study, the values of D (the flexural rigidity of the plate), q (uniform load), a (width of the plate) are always used to be 1 since they are used as scaling parameters. As a result of that, the derived solutions for the deflection, moments and shears of the plates are in the form of functions that are polynomials of x & y multiplied by the scaling parameters$a$, q and D raised to some power. The value of b (length of the plate) is set to be equal to the ratio of b/a and in the cases discussed in this chapter, Poisson's Ratio (ν) is set to have the value 0.3.

5.2 Plates with One Edge Simply Supported and the Opposite Edge Clamped

This section deals with the analysis of four cases that have one edge simply supported and the opposite edge clamped. Figure 9 shows the configuration and the coordinate system for the plates being analyzed in this section. The plate is clamped at $x = -a/2$ and simply supported at $x = a/2$, while the other boundaries at $y = -b/2$ and $y = b/2$ can be either simply supported, clamped or free.

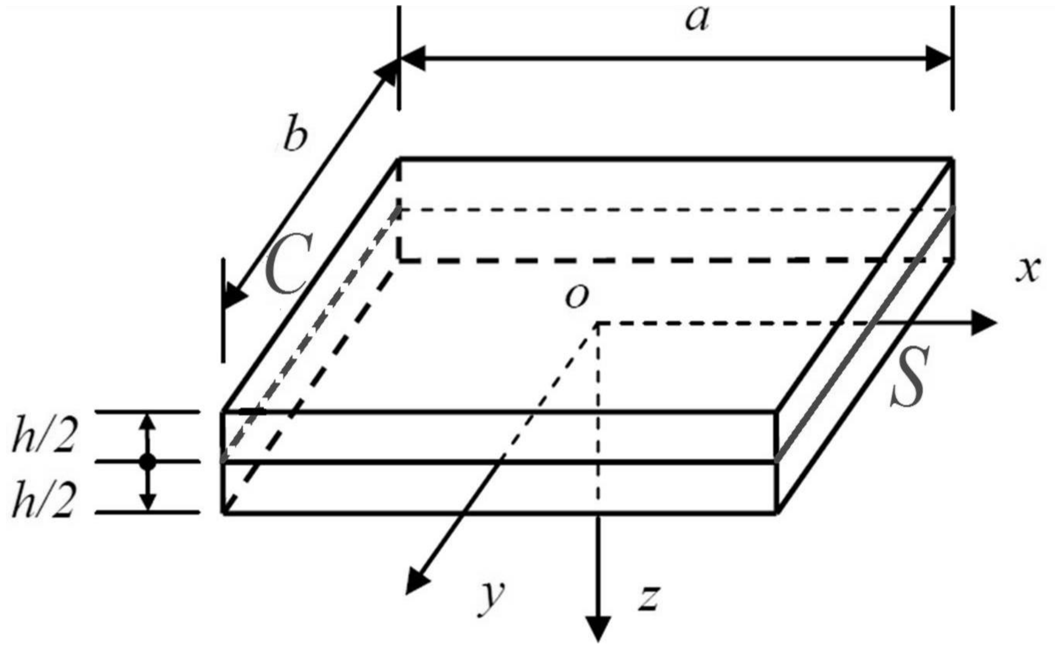

Figure 9 Configuration of Plates with One Edge Simply Supported and the Opposite Edge Clamped

5.2.1 Plate with Two Adjacent Edges Simply Supported and the Other Two Clamped (SCSC)

This plate case was selected to be the starting case in this section because Timoshenko [1] provided some results for this case. This case boundaries are selected to be as follows: clamped at $x = -a/2$ & $y = -b/2$ and simply supported at $x = a/2$ & $y = b/2$. These boundaries provide the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (w_y)_{y=-b/2} = (\frac{\partial w}{\partial y})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \qquad (5.1)$$

In this study, this case was solved with both the Galerkin method and the Ritz method using all the BCs to be satisfied.

The starting assumed function of deflection for the solution was used to be:

$$w = \sum_{i=0}^{n}\sum_{j=0}^{n} C_{i,j}x^i y^j \qquad (5.2)$$

This function does not satisfy any of the boundary conditions, and they are left for Mathematica software to satisfy them. Also it does not provide symmetric solution since the plate is not symmetric in any direction.

In this study, it has been able to derive solutions up to $n = 12$. Timoshenko [1] did not provide full solution for this plate case but he provided some values of moments and deflections. These result are compared with the derived results in Table 17 (values shaded with red color has % difference larger than 10%).

Table 17 shows a little bit high difference between the derived solutions and Timoshenko's solution. This difference is not due to errors in the results provided by Timoshenko, but because Timoshenko did not specify where is the exact location of the provided values. For example, it is mentioned in Timoshenko's book, "Calculations show that the numerically largest moment is produced near the mid-point of the long side of the plate. The values of this clamping moment prove to be $-0.1180qb^2$" *(or* $-0.0295qa^2$) "for $b/a = 0.5$ and $-0.0694qb^2$" *(or* $-0.0694qa^2$) "for $b/a = 1.0$. Etc.". So, he mentioned that these values are near the point $(0, -\frac{b}{2})$ and did not mention the exact location, while the results provided by the research set the results at the point $(0, -\frac{b}{2})$ exactly. Hence, another comparison table were done between the derived solutions and FEM solution derived using COMSOL software. This comparison is in Table 18 and it shows negligible difference between the results.

Table 17 Comparison of results for uniformly loaded rectangular SCSC plate (Galerkin – G, Ritz – R & Timoshenko - T) for ($\nu = 0.3$), $n = 12$

$n = 12$						
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)} * qa^2$	$M_{y(0,-\frac{b}{2})} * qa^2$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	G			-0.0303296		
	T			-0.0295		
	% Diff.			-2.81%		
1	G	0.00210364	-0.0677404	-0.0678726	0.0304254	0.0304285
	T	0.00230	-0.0694	-0.0694	0.034	0.034
	% Diff.	8.54%	2.39%	2.20%	10.51%	10.50%
$n = 12$						
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)} * qa^2$	$M_{y(0,-\frac{b}{2})} * qa^2$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	R			-0.0280841		
	T			-0.0295		
	% Diff.			4.80%		
1	R	0.0021037	-0.0676891	-0.0676151	0.0304412	0.0304404
	T	0.00230	-0.0694	-0.0694	0.034	0.034
	% Diff.	8.53%	2.47%	2.57%	10.47%	10.47%

Table 18 Comparison of results for uniformly loaded rectangular SCSC plate (Galerkin – G, Ritz – R & COMSOL) for ($\nu = 0.3$), $n = 12$

$n = 12$						
b/a	Method	$w_{(0,0)}$ $*\frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)}$ $*qa^2$	$M_{y(0,-\frac{b}{2})}$ $*qa^2$	$M_{x(0,0)}$ $*qa^2$	$M_{y(0,0)}$ $*qa^2$
0.5	G	0.00029283	-0.0201339	-0.0303296	0.00624311	0.0146149
	COMSOL	0.00029463	-0.0199144	-0.0298247	0.00618733	0.01460106
	% Diff.	0.61%	-1.10%	-1.69%	-0.90%	-0.09%
0.75	G	0.00106019	-0.0425907	-0.0525235	0.0172099	0.0251479
	COMSOL	0.00106613	-0.0432101	-0.0533025	0.0172738	0.02521358
	% Diff.	0.56%	1.43%	1.46%	0.37%	0.26%
1	G	0.00210364	-0.0677404	-0.0678726	0.0304254	0.0304285
	COMSOL	0.00211386	-0.068363	-0.0683603	0.03051387	0.03051385
	% Diff.	0.48%	0.91%	0.71%	0.29%	0.28%
1.333	G	0.00335123	-0.0944163	-0.0767072	0.0448045	0.0306366
	COMSOL	0.00336363	-0.0946876	-0.0767669	0.04479184	0.03068876
	% Diff.	0.37%	0.29%	0.08%	-0.03%	0.17%
2	G	0.00468565	-0.121143	-0.0788988	0.0590626	0.0249673
	COMSOL	0.00469482	-0.1190425	-0.0795376	0.05832411	0.02472873
	% Diff.	0.20%	-1.76%	0.80%	-1.27%	-0.96%
$n = 12$						
b/a	Method	$w_{(0,0)}$ $*\frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)}$ $*qa^2$	$M_{y(0,-\frac{b}{2})}$ $*qa^2$	$M_{x(0,0)}$ $*qa^2$	$M_{y(0,0)}$ $*qa^2$
0.5	R	0.00029493	-0.0194479	-0.0280841	0.00677805	0.0150221
	COMSOL	0.00029463	-0.0199144	-0.0298247	0.00618733	0.01460106
	% Diff.	-0.10%	2.34%	5.84%	-9.55%	-2.88%
0.75	R	0.00106057	-0.0443668	-0.0547581	0.0173826	0.02536
	COMSOL	0.00106613	-0.0432101	-0.0533025	0.0172738	0.02521358
	% Diff.	0.52%	-2.68%	-2.73%	-0.63%	-0.58%
1	R	0.0021037	-0.0676891	-0.0676151	0.0304412	0.0304404
	COMSOL	0.00211386	-0.068363	-0.0683603	0.03051387	0.03051385
	% Diff.	0.48%	0.99%	1.09%	0.24%	0.24%
1.333	R	0.00335077	-0.0937422	-0.0760649	0.0446762	0.0306181
	COMSOL	0.00336363	-0.0946876	-0.0767669	0.04479184	0.03068876
	% Diff.	0.38%	1.00%	0.91%	0.26%	0.23%
2	R	0.00468371	-0.118098	-0.0779803	0.0582892	0.0247455
	COMSOL	0.00469482	-0.1190425	-0.0795376	0.05832411	0.02472873
	% Diff.	0.24%	0.79%	1.96%	0.06%	-0.07%

5.2.2 Plate with Two Opposite Edges Free, One Simply Supported and One Clamped (SCFF)

Starting from this SCFF plate case and to the end of this chapter, all the plate cases do not have available solution in Timoshenko book [1]. Therefore, the results are compared with FEM solutions derived using COMSOL software. In this case, the plate was given the following boundaries, clamped at the boundaries, $x = -a/2$, simply supported at $x = a/2$ and free at $y = -b/2$ & $y = b/2$. These boundaries provide the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\,\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\,\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \ \& \ (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\,\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\,\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \ \& \ (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\,\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0 \quad (5.3)$$

In this thesis, this case has been solved with both the Galerkin and the Ritz method by using all the equations of BCs. It has been solved also another time using the Ritz method by just satisfying the essential BCs at the clamped and simply supported edges (3 BCs).

The starting function of deflection in all the solutions was assumed to be:

$$w = \sum_{i=0}^{2n} \sum_{j=0}^{n} C_{i,j} x^i y^{2j} \qquad (5.4)$$

This function does not satisfy any of the BCs by itself but it assure a symmetric solution for the plate around the x axis. The boundary conditions are satisfied by the help of Mathematica software.

Since there is not any available Timoshenko's solution for this case, the results were compared with the FEM solution derived by COMSOL software for 4 b/a ratios, at values of n starting from $n = 1$ to $n = 6$.

Table 19 Comparison of results for uniformly loaded rectangular SCFF plate (Galerkin – G, Ritz 'satisfying all BCs' – R1 & COMSOL) for ($\nu = 0.3$), $n = 6$

$n = 6$						
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)} * qa^2$	$w_{(0,-\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	G	0.0030738	-0.0217066	0.00329529	0.0421539	0.00331104
	COMSOL	0.00533893	-0.1288572	0.00573465	0.06059644	0.00801564
	% Diff.	42.43%	83.15%	42.54%	30.44%	58.69%
0.75	G	0.0043925	-0.12181	0.0046525	0.0536435	0.0184389
	COMSOL	0.00521636	-0.1243454	0.00583567	0.06056043	0.0119751
	% Diff.	15.79%	2.04%	20.27%	11.42%	-53.98%
1	G	0.00440087	-0.214001	0.00460385	0.0601502	0.0199868
	COMSOL	0.00515194	-0.1229616	0.0059012	0.06072795	0.01482591
	% Diff.	14.58%	-74.04%	21.98%	0.95%	-34.81%
1.333	G	0.00475721	-0.429422	0.00441237	0.0629535	0.0307208
	COMSOL	0.00513034	-0.1232792	0.00593997	0.06113004	0.01710607
	% Diff.	7.27%	-248.33%	25.72%	-2.98%	-79.59%
$n = 6$						
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)} * qa^2$	$w_{(0,-\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	R1	0.0030739	-0.0219033	0.00329529	0.0421866	0.00333832
	COMSOL	0.00533893	-0.1288572	0.00573465	0.06059644	0.00801564
	% Diff.	42.42%	83.00%	42.54%	30.38%	58.35%
0.75	R1	0.00420385	-0.0953604	0.00440653	0.0416649	0.0145848
	COMSOL	0.00521636	-0.1243454	0.00583567	0.06056043	0.0119751
	% Diff.	19.41%	23.31%	24.49%	31.20%	-21.79%
1	R1	0.00439542	-0.131444	0.00448291	0.054345	0.0187172
	COMSOL	0.00515194	-0.1229616	0.0059012	0.06072795	0.01482591
	% Diff.	14.68%	-6.90%	24.03%	10.51%	-26.25%
1.333	R1	0.00467282	-0.129434	0.00438938	0.0635944	0.0238012
	COMSOL	0.00513034	-0.1232792	0.00593997	0.06113004	0.01710607
	% Diff.	8.92%	-4.99%	26.10%	-4.03%	-39.14%

Table 19 shows a comparison of the derived solutions using the Galerkin and the Ritz method (satisfying all BCs) for $n = 6$ with the COMSOL solution (values shaded with red have % difference larger than 10%).

It is very clear from the Table 19 that neither the Galerkin method nor the Ritz method solutions agree with the FEM solution if try to satisfy all the BCs. This high difference will totally vanish if one used the Ritz method and just applied the essential BCs. Table 20 compares the Ritz solution (only essential BCs applied) with the COMSOL solution. The table shows an excellent agreement.

Table 20 Comparison of results for uniformly loaded rectangular SCFF plate (Ritz 'satisfying only essential BCs' – R2 & COMSOL) for ($\nu = 0.3$), $n = 6$

$n = 6$						
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)}$ $* qa^2$	$w_{(0,-\frac{b}{2})}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$
0.5	R2	0.00532742	-0.125135	0.00571764	0.0604804	0.00829878
	COMSOL	0.00533893	-0.1288572	0.00573465	0.06059644	0.00801564
	% Diff.	0.22%	2.89%	0.30%	0.19%	-3.53%
0.75	R2	0.00520822	-0.122843	0.00581688	0.0603316	0.0120099
	COMSOL	0.00521636	-0.1243454	0.00583567	0.06056043	0.0119751
	% Diff.	0.16%	1.21%	0.32%	0.38%	-0.29%
1	R2	0.00514538	-0.12176	0.0058811	0.0603981	0.014576
	COMSOL	0.00515194	-0.1229616	0.0059012	0.06072795	0.01482591
	% Diff.	0.13%	0.98%	0.34%	0.54%	1.69%
1.333	R2	0.00512486	-0.122663	0.0059182	0.0608996	0.016917
	COMSOL	0.00513034	-0.1232792	0.00593997	0.06113004	0.01710607
	% Diff.	0.11%	0.50%	0.37%	0.38%	1.11%

5.2.3 Plate with Two Adjacent Edges Simply Supported, One Clamped and One Free (SCSF)

The analysis, results, comparison tables and outcomes of SCSF plate are very much similar to those discussed in the previous SCFF case. For the current case, the plate was chosen to be clamped at edge $x = -a/2$, simply supported at $x = a/2$ & $y = b/2$ and free at $y = -b/2$. These boundary conditions are the results of these boundaries:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \ \& \ (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \tag{5.5}$$

As in SCSF case, this case has been solved with both the Galerkin and the Ritz method by using all the equations of BCs. It has been solved also another time using the Ritz method by just satisfying the essential BCs at the clamped and simply supported edges (4 BCs).

The assumed starting function of deflection was used to be:

$$w = \sum_{i=0}^{n}\sum_{j=0}^{n} C_{i,j}x^i y^j \tag{5.6}$$

This function does not satisfy any of the BCs by itself and does not provide symmetric solution since the plate is not symmetric.

The derived results were compared with the FEM solution derived by COMSOL software for 5 b/a ratios, at values of n starting from $n = 1$ to $n = 10$. Table 21 shows a comparison of the derived solutions using the Galerkin and the Ritz method (satisfying all BCs) for $n = 10$ with the COMSOL solution (values shaded with red have % difference larger than 10%).

Table 21 Comparison of results for uniformly loaded rectangular SCSF plate (Galerkin – G, Ritz 'satisfying all BCs' – R1 & COMSOL) for ($\nu = 0.3$), $n = 10$

$n = 10$						
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)}$ $* qa^2$	$w_{(0,-\frac{b}{2})}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$
0.5	G	0.00200876	-0.0651112	0.00345445	0.0338481	0.030551
	COMSOL	0.00236464	-0.0697282	0.00430099	0.02992335	0.01757571
	% Diff.	15.05%	6.62%	19.68%	-13.12%	-73.83%
0.75	G	0.00281635	-0.0889549	0.00408288	0.0402686	0.0286392
	COMSOL	0.00331663	-0.0901668	0.00542787	0.04216664	0.02299388
	% Diff.	15.08%	1.34%	24.78%	4.50%	-24.55%
1	G	0.00330484	-0.0942858	0.00406348	0.0147506	0.0176231
	COMSOL	0.0039599	-0.103578	0.00583001	0.04987696	0.02449866
	% Diff.	16.54%	8.97%	30.30%	70.43%	28.07%
1.333	G	0.0039675	-0.125674	0.00363582	0.0444706	0.0227801
	COMSOL	0.00451503	-0.114558	0.00595768	0.05601558	0.02383773
	% Diff.	12.13%	-9.70%	38.97%	20.61%	4.44%
2	G	0.00478922	-0.074719	0.00317809	0.0596233	0.0203197
	COMSOL	0.00502783	-0.123673	0.00596075	0.06107648	0.02108219
	% Diff.	4.75%	39.58%	46.68%	2.38%	3.62%

$n = 10$						
b/a	Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)}$ $* qa^2$	$w_{(0,-\frac{b}{2})}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$
0.5	R1	0.00187042	-0.0652306	0.00329212	0.0226307	0.0127798
	COMSOL	0.00236464	-0.0697282	0.00430099	0.02992335	0.01757571
	% Diff.	20.90%	6.45%	23.46%	24.37%	27.29%
0.75	R1	0.00278999	-0.105276	0.00406123	0.0425511	0.0313868
	COMSOL	0.00331663	-0.0901668	0.00542787	0.04216664	0.02299388
	% Diff.	15.88%	-16.76%	25.18%	-0.91%	-36.50%
1	R1	0.00350025	-0.100673	0.00423566	0.0510737	0.0313763
	COMSOL	0.0039599	-0.103578	0.00583001	0.04987696	0.02449866
	% Diff.	11.61%	2.80%	27.35%	-2.40%	-28.07%
1.333	R1	0.00408846	-0.117138	0.00398936	0.0506915	0.0239137
	COMSOL	0.00451503	-0.114558	0.00595768	0.05601558	0.02383773
	% Diff.	9.45%	-2.25%	33.04%	9.50%	-0.32%
2	R1	0.00477096	-0.12697	0.00295582	0.0542072	0.0236734
	COMSOL	0.00502783	-0.123673	0.00596075	0.06107648	0.02108219
	% Diff.	5.11%	-2.67%	50.41%	11.25%	-12.29%

The big difference in results is obvious in Table 21. However, as in the previous case, if only essential BCs are applied with the Ritz method, then the difference in results disappears. Table 22 shows that clearly.

Table 22 Comparison of results for uniformly loaded rectangular SCSF plate (Ritz 'satisfying only essential BCs' – R2 & COMSOL) for ($\nu = 0.3$), $n = 10$

$n = 10$						
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)} * qa^2$	$w_{(0,-\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	R2	0.0023396	-0.0682561	0.00424765	0.0297049	0.0174561
	COMSOL	0.00236464	-0.0697282	0.00430099	0.02992335	0.01757571
	% Diff.	1.06%	2.11%	1.24%	0.73%	0.68%
0.75	R2	0.00329872	-0.0890601	0.00538585	0.0421923	0.023321
	COMSOL	0.00331663	-0.0901668	0.00542787	0.04216664	0.02299388
	% Diff.	0.54%	1.23%	0.77%	-0.06%	-1.42%
1	R2	0.00394376	-0.103073	0.00579829	0.0494289	0.023999
	COMSOL	0.0039599	-0.103578	0.00583001	0.04987696	0.02449866
	% Diff.	0.41%	0.49%	0.54%	0.90%	2.04%
1.333	R2	0.00450145	-0.114032	0.00593088	0.0556105	0.023488
	COMSOL	0.00451503	-0.114558	0.00595768	0.05601558	0.02383773
	% Diff.	0.30%	0.46%	0.45%	0.72%	1.47%
2	R2	0.00501715	-0.125847	0.00593281	0.0606689	0.0208838
	COMSOL	0.00502783	-0.123673	0.00596075	0.06107648	0.02108219
	% Diff.	0.21%	-1.76%	0.47%	0.67%	0.94%

5.2.4 Plate with Two Adjacent Edges Clamped, One Simply Supported and One Free (SCCF)

The SCCF plate case is also similar to the previous two cases in terms of procedure, results and outcomes. It just differs from the SCSF by one of the BCs. For this case, the plate is clamped at edges $x = -a/2$ & $y = b/2$, simply supported edge at $x = a/2$ and free edge at $y = -b/2$, which provide the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\,\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \;\&\; (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-\upsilon) \frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (w_y)_{y=b/2} = (\frac{\partial w}{\partial y})_{y=b/2} = 0 \tag{5.7}$$

Similar to the previous two cases, this case has been solved with both the Galerkin and the Ritz method by applying all the equations of BCs. It has been also solved one more time using the Ritz method by just satisfying the essential BCs at the clamped and simply supported edges (5 BCs).

The assumed starting function of deflection was used to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} x^i y^j \tag{5.8}$$

It is clear that the function does not satisfy any of the BCs by itself and does not provide symmetric solution since the plate is not symmetric.

The solutions were compared with the FEM solution derived by COMSOL software for 4 b/a ratios, at values of n starting from $n = 1$ to $n = 10$. Table 23 shows a comparison of the derived solutions using the Galerkin and the Ritz method (satisfying all BCs) for $n = 10$ with the COMSOL solution (values shaded with red have % difference larger than 10%).

Table 23 Comparison of results for uniformly loaded rectangular SCCF plate (Galerkin – G, Ritz 'satisfying all BCs' – R1 & COMSOL) for ($\nu = 0.3$), $n = 10$

$n = 10$							
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)} * qa^2$	$M_{y(0,\frac{b}{2})} * qa^2$	$w_{(0,-\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	G	0.00097287	-0.0376744	-0.0697736	0.00221628	0.0172922	0.0102494
	COMSOL	0.00111285	-0.0357744	-0.0647989	0.00271845	0.01126621	0.0011251
	% Diff.	12.58%	-5.31%	-7.68%	18.47%	-53.49%	-810.98%
0.75	G	0.00191251	-0.114022	-0.11053	0.00366394	0.0278276	0.019377
	COMSOL	0.00223662	-0.0631372	-0.0752201	0.00467066	0.02738217	0.01370542
	% Diff.	14.49%	-80.59%	-46.94%	21.55%	-1.63%	-41.38%
1	G	0.00273824	-0.0989009	-0.083539	0.00400114	0.0364814	0.0269414
	COMSOL	0.00312077	-0.0842041	-0.0776643	0.0055467	0.03935283	0.02095965
	% Diff.	12.26%	-17.45%	-7.56%	27.86%	7.30%	-28.54%
1.333	G	0.00358023	-0.13119	-0.0721395	0.00378302	0.0447133	0.0255419
	COMSOL	0.00396004	-0.1030423	-0.0787928	0.00589983	0.04970477	0.02397334
	% Diff.	9.59%	-27.32%	8.44%	35.88%	10.04%	-6.54%
$n = 10$							
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)} * qa^2$	$M_{y(0,\frac{b}{2})} * qa^2$	$w_{(0,-\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	R1	0.00094193	-0.0207378	-0.0549753	0.00220824	0.0109804	0.00173679
	COMSOL	0.00111285	-0.0357744	-0.0647989	0.00271845	0.01126621	0.0011251
	% Diff.	15.36%	42.03%	15.16%	18.77%	2.54%	-54.37%
0.75	R1	0.00160079	-0.0441818	-0.0545618	0.00283875	0.0218797	0.0147826
	COMSOL	0.00223662	-0.0631372	-0.0752201	0.00467066	0.02738217	0.01370542
	% Diff.	28.43%	30.02%	27.46%	39.22%	20.10%	-7.86%
1	R1	0.00257477	-0.0695216	-0.0742847	0.00354288	0.0358355	0.0247336
	COMSOL	0.00312077	-0.0842041	-0.0776643	0.0055467	0.03935283	0.02095965
	% Diff.	17.50%	17.44%	4.35%	36.13%	8.94%	-18.01%
1.333	R1	0.00358918	-0.118441	-0.0744984	0.00411253	0.0470803	0.0273174
	COMSOL	0.00396004	-0.1030423	-0.0787928	0.00589983	0.04970477	0.02397334
	% Diff.	9.37%	-14.94%	5.45%	30.29%	5.28%	-13.95%

As in the previous two cases, the large difference in results is obvious in Table 23. In Table 24, only the essential BCs are applied with the Ritz method and it is seen that the difference in results becomes negligible.

Table 24 Comparison of results for uniformly loaded rectangular SCCF plate (Ritz 'satisfying only essential BCs' – R2 & COMSOL) for ($\nu = 0.3$), $n = 10$

n=10							
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(-\frac{a}{2},0)} * qa^2$	$M_{y(0,\frac{b}{2})} * qa^2$	$w_{(0,-\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$
0.5	R2	0.00110461	-0.0355566	-0.0645044	0.00269144	0.0111301	0.00114197
	COMSOL	0.00111285	-0.0357744	-0.0647989	0.00271845	0.01126621	0.0011251
	% Diff.	0.74%	0.61%	0.45%	0.99%	1.21%	-1.50%
0.75	R2	0.00222839	-0.0622963	-0.076396	0.00463437	0.0275819	0.01389
	COMSOL	0.00223662	-0.0631372	-0.0752201	0.00467066	0.02738217	0.01370542
	% Diff.	0.37%	1.33%	-1.56%	0.78%	-0.73%	-1.35%
1	R2	0.00311364	-0.0840574	-0.0769961	0.00551047	0.0394508	0.021153
	COMSOL	0.00312077	-0.0842041	-0.0776643	0.0055467	0.03935283	0.02095965
	% Diff.	0.23%	0.17%	0.86%	0.65%	-0.25%	-0.92%
1.333	R2	0.00395357	-0.101329	-0.0779546	0.00586849	0.0497927	0.0240911
	COMSOL	0.00396004	-0.1030423	-0.0787928	0.00589983	0.04970477	0.02397334
	% Diff.	0.16%	1.66%	1.06%	0.53%	-0.18%	-0.49%

5.3 Plates with One Edge Simply Supported and the Opposite Edge Free

In this section, the thesis discusses the second type of cases with unsymmetrical opposite edges, which includes three plate cases with one edge simply supported and the opposite edge free. Figure 10 generalize the configuration and the coordinate system for the plates discussed in this section. The plate has a free edge at $x = -a/2$ and simply supported edge at $x = a/2$, while the other boundaries at $y = -b/2$ and $y = b/2$ can be either simply supported, clamped or free.

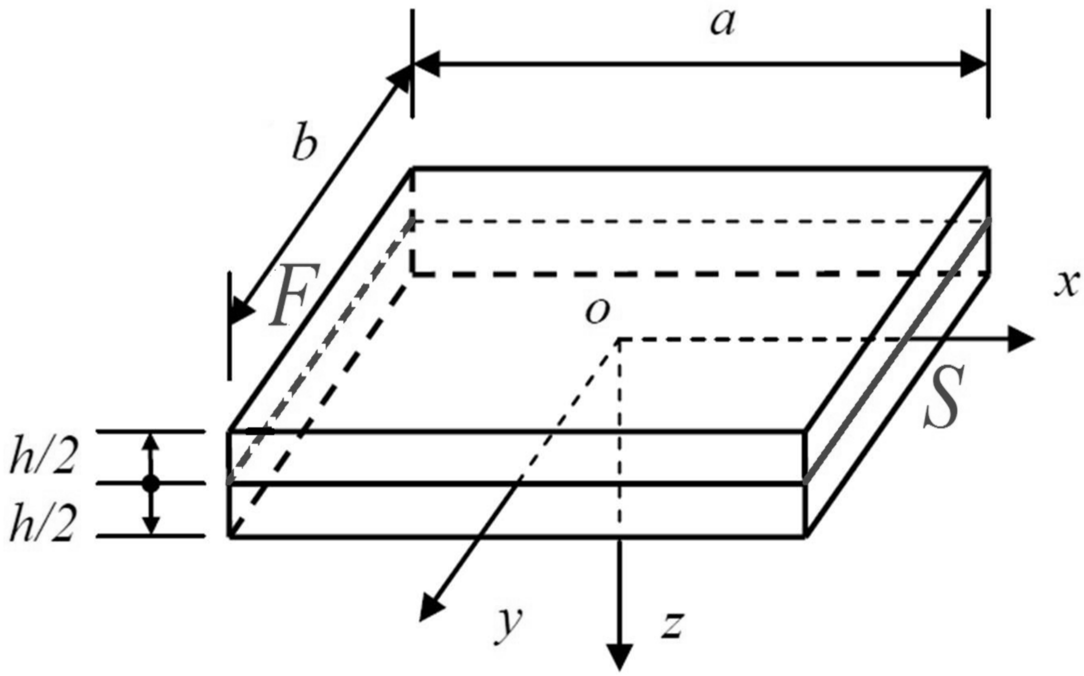

Figure 10 Configuration of Plates with One Edge Simply Supported and the Opposite Edge Free

5.3.1 Plate with Two Adjacent Edges Simply Supported and the Other Two Free (SFSF)

The first plate case analyzed in this section is the SFSF case. This case boundaries are selected to be as follows: free at $x = -a/2$ & $y = -b/2$ and simply supported at $x = a/2$ & $y = b/2$. These boundaries provide the following boundary conditions:

$$(M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0 \ \& \ (V_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + (2-v)\frac{\partial^2 w}{\partial x \partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \ \& \ (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \qquad (5.9)$$

In this study, this case was not able to be solved with the Galerkin method, but it was successfully solved with the Ritz method with two different boundary conditions applied. It was solved one time by applying all the BCs and another time by just applying the BCs at the simply supported edges (4 BCs).

The starting assumed function of deflection for the solution was in the general form:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} x^i y^j \tag{5.10}$$

In this study, solutions of 5 b/a ratios were derived for values of n up to $n = 10$. The derived solutions are compared with the FEM solution derived using COMSOL software in Table 25. (values shaded with red color has % difference larger than 10%).

Table 25 Comparison of results for uniformly loaded rectangular SFSF plate (Ritz 'satisfying all BCs' – R1, Ritz 'satisfying only BCs at S edges' – R2 & COMSOL) for ($\nu = 0.3$), $n = 4$

$n = 10$								
b/a	Method	$w_{(-\frac{a}{2},-\frac{b}{2})} * \frac{qa^4}{D}$	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,-\frac{b}{2})} * qa^2$	$M_{y(-\frac{a}{2},0)} * qa^2$
0.5	R1	0.0446429	0.0150869	0.0354584	0.0232631	0.0585143	0.0419341
	COMSOL	0.04550873	0.01538421	0.03760316	0.02547033	0.06013346	0.03794978
	% Diff.	1.90%	1.93%	5.70%	8.67%	2.69%	-10.50%
0.75	R1	0.100446	0.0324491	0.0587305	0.049606	0.0970054	0.0771082
	COMSOL	0.1019516	0.03291172	0.05846859	0.04815429	0.09425657	0.07681959
	% Diff.	1.48%	1.41%	-0.45%	-3.01%	-2.92%	-0.38%
1	R1	0.178572	0.0570465	0.0791454	0.0753621	0.119207	0.124859
	COMSOL	0.18085863	0.05766441	0.0727408	0.07274092	0.11747831	0.11795636
	% Diff.	1.26%	1.07%	-8.80%	-3.60%	-1.47%	-5.85%
1.333	R1	0.31746	0.102355	0.0739589	0.0929355	0.134297	0.171494
	COMSOL	0.32093692	0.10364081	0.08562417	0.10381157	0.13619936	0.16832797
	% Diff.	1.08%	1.24%	13.62%	10.48%	1.40%	-1.88%
2	R1	0.714286	0.241513	0.103348	0.14799	0.154424	0.23631
	COMSOL	0.72049701	0.24368423	0.10200622	0.14953346	0.15178801	0.24126127
	% Diff.	0.86%	0.89%	-1.32%	1.03%	-1.74%	2.05%

$n = 10$							
b/a	Method	$w_{(-\frac{a}{2},-\frac{b}{2})} * \frac{qa^4}{D}$	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,-\frac{b}{2})} * qa^2$	$M_{y(-\frac{a}{2},0)} * qa^2$
0.5	R2	0.0446429	0.0151053	0.0372558	0.0255924	0.0602242	0.0376316
	COMSOL	0.04550873	0.01538421	0.03760316	0.02547033	0.06013346	0.03794978
	% Diff.	1.90%	1.81%	0.92%	-0.48%	-0.15%	0.84%
0.75	R2	0.100446	0.0324766	0.0590636	0.0498483	0.0931439	0.0793606
	COMSOL	0.1019516	0.03291172	0.05846859	0.04815429	0.09425657	0.07681959
	% Diff.	1.48%	1.32%	-1.02%	-3.52%	1.18%	-3.31%
1	R2	0.178571	0.0570109	0.0726876	0.0726876	0.117795	0.117795
	COMSOL	0.18085863	0.05766441	0.0727408	0.07274092	0.11747831	0.11795636
	% Diff.	1.26%	1.13%	0.07%	0.07%	-0.27%	0.14%
1.333	R2	0.31746	0.102642	0.0886371	0.10501	0.141159	0.165596
	COMSOL	0.32093692	0.10364081	0.08562417	0.10381157	0.13619936	0.16832797
	% Diff.	1.08%	0.96%	-3.52%	-1.15%	-3.64%	1.62%
2	R2	0.714286	0.241685	0.10237	0.149025	0.150531	0.240931
	COMSOL	0.72049701	0.24368423	0.10200622	0.14953346	0.15178801	0.24126127
	% Diff.	0.86%	0.82%	-0.36%	0.34%	0.83%	0.14%

Table 25 shows good agreement between the derived solutions and the FEM solution. It is noted that when all the BCs are tried to be satisfied (R1 solution), some of the results are a little bit far from the FEM solutions. However, if some of the non-essential BCs are removed from the satisfied BCs list, then the solution provides excellent agreement with the FEM solution.

5.3.2 Plate with Two Adjacent Edges Free, One Simply Supported and One Clamped (SFCF)

SFCF case is one of the most complicated cases because of the variety of type of boundaries that it contains and un-symmetricity. For this case, the boundaries were selected to be as follows: free at $x = -a/2$ & $y = -b/2$, simply supported at $x = a/2$ and clamped at $y = b/2$, which give the following boundary conditions:

$$(M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0 \; \& \; (V_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + (2-v)\frac{\partial^2 w}{\partial x \partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \; \& \; (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (w_y)_{y=b/2} = (\frac{\partial w}{\partial y})_{y=b/2} = 0 \tag{5.11}$$

In this study, no solution could be obtained if all the BC equations applied, neither using the Galerkin method nor the Ritz method. Therefore, it has been tried to derive solution using the Ritz method by just applying the essential boundary conditions (3 BCs).

The starting assumed function of deflection for the solution was in the general form:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} x^i y^j \tag{5.12}$$

Solutions for 4 b/a ratios were derived for values of n starting from $n = 1$ to $n = 10$. The derived solutions showed good agreement with the FEM solution derived using COMSOL software when $n = 10$. Table 26 shows this comparison (values shaded with red color has % difference larger than 10%).

Table 26 provides good agreement of the results at all the compared values, except for the value of M_x at the center of the plate for b/a ratio equal to 1. This difference which is around 18% difference is quite huge and needs further analysis to go over the reasons of it. As in CCCF and CCSF cases, one possible reason for that difference is the small magnitude of M_x compared to the magnitude of M_y at the center the plate with this ratio. Very small values tend to have higher error because they are near to zero. This high difference makes finding exact values of M_x for this case not very important and not critical in design. Moreover, if the values of M_x for b/a ratios smaller than 1 are compared with M_x for b/a ratios larger than 1, it will be noticed that the magnitude of the moment is turning from negative to positive value, which means that around $b/a = 1$, the magnitude of $M_x \approx 0$ and that explains the high difference in results.

Table 26 Comparison of results for uniformly loaded rectangular SFCF plate (Ritz 'satisfying essential BCs only' – R & COMSOL) for ($\nu = 0.3$), $n = 10$

$n = 10$								
b/a	Method	$w_{(-\frac{a}{2},-\frac{b}{2})} * \frac{qa^4}{D}$	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,-\frac{b}{2})} * qa^2$	$M_{y(0,\frac{b}{2})} * qa^2$
0.75	R	0.0292613	0.00768707	0.0122603	-0.0140664	0.0443477	-0.181097
	COMSOL	0.0294546	0.0077286	0.01205582	-0.0147322	0.04376025	-0.180179
	% Diff.	0.66%	0.54%	-1.70%	4.52%	-1.34%	-0.51%
1	R	0.0713265	0.017994	0.0247918	-0.0038987	0.0729596	-0.253064
	COMSOL	0.07183374	0.01808804	0.02438017	-0.0047736	0.07189887	-0.2533559
	% Diff.	0.71%	0.52%	-1.69%	18.33%	-1.48%	0.12%
1.333	R	0.159585	0.0406043	0.0397169	0.0195096	0.0979649	-0.347708
	COMSOL	0.1606994	0.04080433	0.03846547	0.0179231	0.10161403	-0.3508475
	% Diff.	0.69%	0.49%	-3.25%	-8.85%	3.59%	0.89%
2	R	0.442682	0.122217	0.0618699	0.0723617	0.134557	-0.552317
	COMSOL	0.450898815	0.124478887	0.061302747	0.072763347	0.135953403	-0.559048412
	% Diff.	1.82%	1.82%	-0.93%	0.55%	1.03%	1.20%

5.3.3 Plate with One Edge Simply Supported and the Other Three Edges Free with Two Supports at the Corners (SFFF)

SFFF case is a unique case among the previously discussed cases since two supports (pins or columns) should be provided at the corners opposite to the simply supported edge to make the plate stable. If these supports are not provided, then the plate will fail under the load. The simply supported edge is used to be at $x = a/2$ while the rest three edges at $x = -a/2$, $y = -b/2$ & $y = b/2$ are free. The corner supports are at points $(-a/2, -b/2)$ and $(-a/2, b/2)$.This configuration is shown in Figure 11.

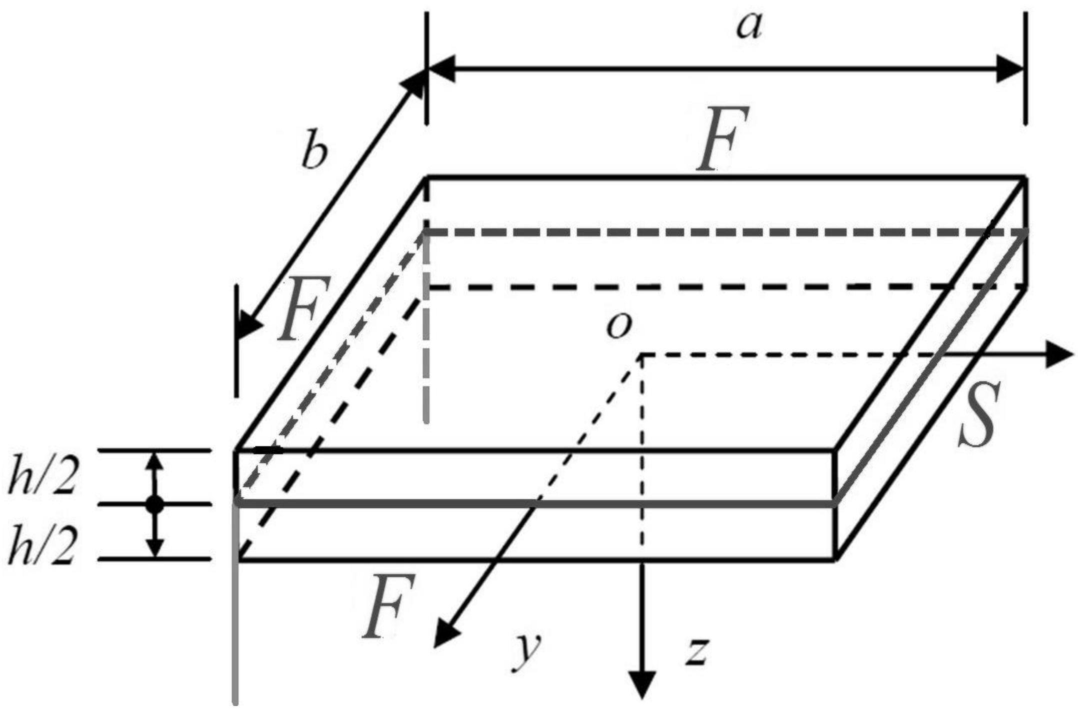

Figure 11 Configuration of Plates with One Edge Simply Supported and the Other Three Edges Free with Two Supports at the Corners (SFFF)

The previous configuration gives the following 10 boundary conditions:

$$(M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0 \; \& \; (V_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + (2-v)\frac{\partial^2 w}{\partial x \partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \; \& \; (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \; \& \; (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0$$

$$(w)_{x=-a/2,y=-b/2} = 0 \qquad \& \qquad (w)_{x=-a/2,y=b/2} = 0 \qquad (5.13)$$

Using both the Galerkin and Ritz methods, no solution could be obtained if all the BC equations applied. However, using the Ritz method, if just the essential BCs are applied (1 at S edge and 1 at each corner = 3 BCs), then great results are obtained.

The starting assumed function of deflection for the solution was in the general form:

$$w = \sum_{i=0}^{2n} \sum_{j=0}^{n} C_{i,j} x^i y^{2j} \tag{5.14}$$

This assumed function provides symmetric solution around the x axis.

Solutions for values of n starting from $n = 1$ to $n = 6$ were derived and found that at $n = 6$, the results perfectly agrees with the FEM solution derived using COMSOL software. Table 27 shows this comparison for 5 different b/a ratios.

Table 27 Comparison of results for uniformly loaded rectangular SFFF plate (Ritz 'satisfying essential BCs only' – R & COMSOL) for ($\nu = 0.3$), $n = 6$

$n = 6$								
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$w_{(\frac{a}{2},0)} * \frac{qa^4}{D}$	$w_{(0,\frac{b}{2})} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{y(\frac{a}{2},0)} * qa^2$
0.5	R	0.0140762	0.00160285	0.0148769	0.123081	0.016802	0.128956	0.0535569
	COMSOL	0.01411429	0.00167356	0.0149261	0.1229872	0.016408	0.128629	0.05333204
	% Diff.	0.27%	4.23%	0.33%	-0.08%	-2.40%	-0.25%	-0.42%
0.75	R	0.0150948	0.00596665	0.0155037	0.120445	0.0380109	0.133925	0.0890734
	COMSOL	0.01515699	0.00610933	0.01558746	0.120462	0.03776502	0.13346702	0.08863465
	% Diff.	0.41%	2.34%	0.54%	0.01%	-0.65%	-0.34%	-0.50%
1	R	0.0183434	0.014993	0.0161698	0.117967	0.0627947	0.139291	0.126083
	COMSOL	0.01844237	0.01523799	0.01628693	0.11787959	0.06278719	0.13893797	0.12556977
	% Diff.	0.54%	1.61%	0.72%	-0.07%	-0.01%	-0.25%	-0.41%
1.333	R	0.0275543	0.0364398	0.0169372	0.115686	0.0952356	0.145294	0.172487
	COMSOL	0.02773548	0.03687782	0.01708849	0.11546817	0.09530199	0.14550723	0.17170072
	% Diff.	0.65%	1.19%	0.89%	-0.19%	0.07%	0.15%	-0.46%
2	R	0.0653225	0.115834	0.0178465	0.114993	0.143526	0.153436	0.240618
	COMSOL	0.06580829	0.11689436	0.01803156	0.11493066	0.14420304	0.15346135	0.24070233
	% Diff.	0.74%	0.91%	1.03%	-0.05%	0.47%	0.02%	0.04%

5.4 Plates with One Edge Clamped and the Opposite Edge Free

The last section of this chapter deals with the last type of cases with unsymmetrical opposite edges. This type includes two plate cases with one edge clamped and the opposite edge free. Figure 12 shows the general configuration and the coordinate system for the plates discussed in this section. The plate has a free edge at $x = -a/2$ and clamped edge at $x = a/2$. The other two boundaries at $y = -b/2$ and $y = b/2$ can be either clamped or free.

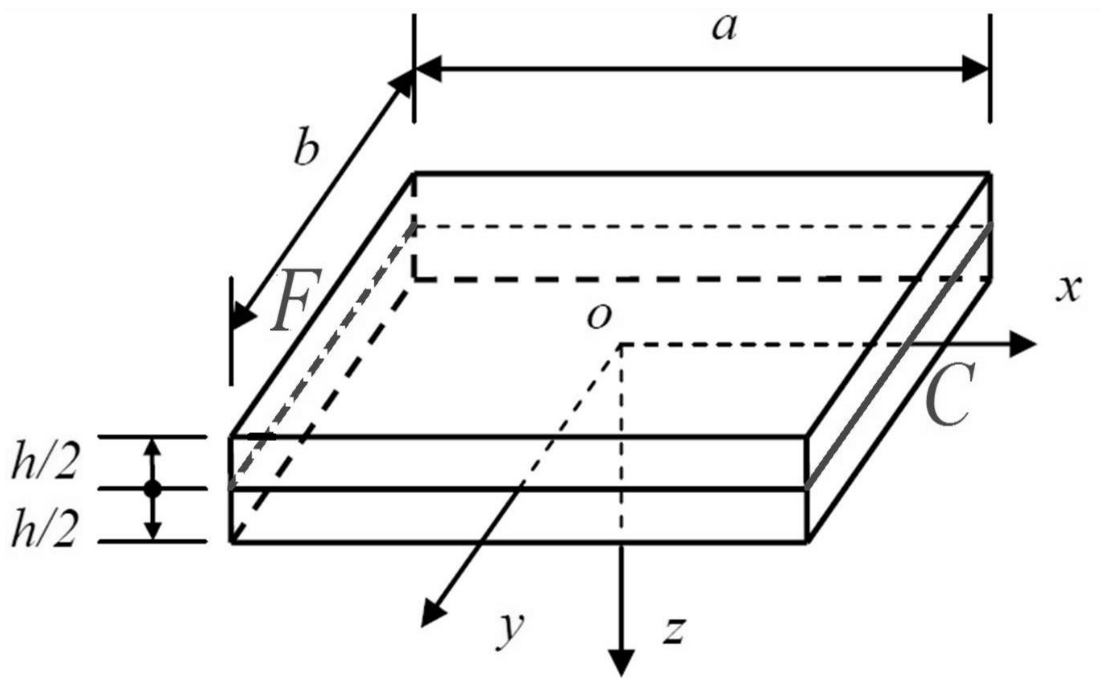

Figure 12 Configuration of Plates with One Edge Clamped and the Opposite Edge Free

5.4.1 Plate with One Edge Clamped and the Other Three Edges Free (CFFF)

Since the latest discussed case is SFFF, now the thesis moves to the analysis of CFFF case. It seems that there is just a small difference between the two cases since just the simply supported edge is replaced with clamped edge. However, the way of analysis and treatment is totally different. In CFFF case, there is no need for the two supports (pins or columns) at the corners opposite to the clamped edge since the clamped edge makes the plate stable. The clamped edge is used to be at $x = -a/2$ while the rest three edges at $x = a/2$, $y = -b/2$ & $y = b/2$ are free. This configuration provides the following boundary conditions:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (w_x)_{x=-a/2} = (\frac{\partial w}{\partial x})_{x=-a/2} = 0$$

$$(M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0 \; \& \; (V_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + (2-v)\frac{\partial^2 w}{\partial x \partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \; \& \; (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \; \& \; (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0 \quad (5.15)$$

As in SFFF case, using both the Galerkin and Ritz methods, no solution could be obtained if all the BC equations applied. However, if Ritz method is used and just the essential BCs are applied (2 BCs at C edge), then accurate results are obtained.

The starting assumed function of deflection for the solution was in the general form:

$$w = \sum_{i=0}^{2n} \sum_{j=0}^{n} C_{i,j} x^i y^{2j} \qquad (5.16)$$

This assumed function provides symmetric solution around the x axis since there are two opposite free edges.

The derived solutions at $n = 6$ for 4 b/a ratios shows great agreement with the FEM solution derived using COMSOL software. Table 28 shows this comparison and it is clear from the table how nice are the results.

Table 28 Comparison of results for uniformly loaded rectangular CFFF plate (Ritz 'satisfying essential BCs only' – R & COMSOL) for (ν=0.3), $n = 6$

$n = 6$					
b/a	Method	$w_{(0,0)}$ $*\frac{qa^4}{D}$	$w_{(\frac{a}{2},0)}$ $*\frac{qa^4}{D}$	$w_{(\frac{a}{2},\frac{b}{2})}$ $*\frac{qa^4}{D}$	$M_{x(0,0)}$ $*qa^2$
0.75	R	0.0459481	0.129674	0.128644	-0.121115
	COMSOL	0.046014	0.1297999	0.12878984	-0.122708
	% Diff.	0.14%	0.10%	0.11%	1.30%
1	R	0.0458308	0.129052	0.127198	-0.123317
	COMSOL	0.04588186	0.1291421	0.12731223	-0.1226357
	% Diff.	0.11%	0.07%	0.09%	-0.56%
1.333	R	0.0456705	0.128554	0.125809	-0.125667
	COMSOL	0.04572071	0.1286177	0.12588996	-0.1237532
	% Diff.	0.11%	0.05%	0.06%	-1.55%
2	R	0.0453086	0.127757	0.124298	-0.126531
	COMSOL	0.04535652	0.12780949	0.12438914	-0.1260943
	% Diff.	0.11%	0.04%	0.07%	-0.35%
b/a	Method	$M_{y(0,0)}$ $*qa^2$	$M_{x(-\frac{a}{2},0)}$ $*qa^2$	$M_{x(0,\frac{b}{2})}$ $*qa^2$	$M_{y(-\frac{a}{2},0)}$ $*qa^2$
0.75	R	-0.0151528	-0.529317	-0.12428	-0.158795
	COMSOL	-0.0151747	-0.542786	-0.1282446	-0.1630012
	% Diff.	0.14%	2.48%	3.09%	2.58%
1	R	-0.0228502	-0.514155	-0.120921	-0.154246
	COMSOL	-0.0236419	-0.5328305	-0.1291817	-0.160001
	% Diff.	3.35%	3.50%	6.39%	3.60%
1.333	R	-0.0317144	-0.518129	-0.129899	-0.155439
	COMSOL	-0.0307392	-0.524631	-0.1291837	-0.1575922
	% Diff.	-3.17%	1.24%	-0.55%	1.37%
2	R	-0.037978	-0.522344	-0.127309	-0.156703
	COMSOL	-0.036747	-0.5153098	-0.1280052	-0.154916
	% Diff.	-3.35%	-1.37%	0.54%	-1.15%

5.4.2 Plate with Two Adjacent Edges Clamped and the Other Two Free (CFCF)

The last case to be discussed in this chapter is the CFCF. The case is similar in difficulty in analysis to the CCFF and SFCF case due to the combinations of variety types of boundaries, especially the clamped and free boundaries which complicate the constraints combination, as well as the un-symmetricity of the plate. For this case, the boundaries were selected to be as follows: free at $x = -a/2$ & $y = -b/2$ and clamped at $x = a/2$ & $y = b/2$, which give the following boundary conditions:

$$(M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0 \ \& \ (V_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + (2-v)\frac{\partial^2 w}{\partial x \partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (w_x)_{x=a/2} = (\frac{\partial w}{\partial x})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \ \& \ (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-v)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (w_y)_{y=b/2} = (\frac{\partial w}{\partial y})_{y=b/2} = 0 \qquad (5.17)$$

In this study, no solution could be obtained if all the BC equations were applied, neither using the Galerkin method nor the Ritz method. Therefore, it has been tried to derive solution using the Ritz method by just applying the essential boundary conditions (4 BCs at clamped edges).

The starting assumed function of deflection for the solution was in the general form:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} x^i y^j \qquad (5.18)$$

Solutions for 5 b/a ratios were derived for values of n starting from $n = 1$ to $n = 10$. The derived solutions showed good agreement with the FEM solution derived using COMSOL software when $n = 10$. Table 29 shows this comparison (values shaded with red color has % difference larger than 10%).

Table 29 Comparison of results for uniformly loaded rectangular CFCF plate (Ritz 'satisfying essential BCs only' – R & COMSOL) for ($\nu = 0.3$), $n = 10$

$n = 10$							
b/a	Method	$w_{(-\frac{a}{2},-\frac{b}{2})}$ $*\frac{qa^4}{D}$	$w_{(0,0)}$ $*\frac{qa^4}{D}$	$M_{x(0,0)}$ $*qa^2$	$M_{y(0,0)}$ $*qa^2$	$M_{x(\frac{a}{2},0)}$ $*qa^2$	$M_{y(0,\frac{b}{2})}$ $*qa^2$
0.5	R	0.00656036	0.00161884	0.00270256	-0.0106513	-0.038194	-0.0838705
	COMSOL	0.00659736	0.00162605	0.00273377	-0.0104577	-0.0388736	-0.0821354
	% Diff.	0.56%	0.44%	1.14%	-1.85%	1.75%	-2.11%
0.75	R	0.0223994	0.00466759	0.00567059	-0.0068665	-0.0775253	-0.112952
	COMSOL	0.02253132	0.00468582	0.00560889	-0.0066831	-0.0814762	-0.1139379
	% Diff.	0.59%	0.39%	-1.10%	-2.74%	4.85%	0.87%
1	R	0.043583	0.00870311	0.00344428	0.0034484	-0.12256	-0.122494
	COMSOL	0.04379919	0.00871677	0.00098946	0.00098945	-0.1309955	-0.1309886
	% Diff.	0.49%	0.16%	Negligible	Negligible	6.44%	6.48%
1.333	R	0.070793	0.0147519	-0.0122079	0.0100766	-0.200798	-0.137805
	COMSOL	0.07111197	0.01479612	-0.0118096	0.01001439	-0.2024633	-0.1447256
	% Diff.	0.45%	0.30%	-3.37%	-0.62%	0.82%	4.78%
2	R	0.104966	0.0259015	-0.0426053	0.0108102	-0.335482	-0.152776
	COMSOL	0.10527236	0.02595926	-0.0415717	0.01100166	-0.3280624	-0.1550387
	% Diff.	0.29%	0.22%	-2.49%	1.74%	-2.26%	1.46%

Table 29 provides good agreement of the results at all the compared values, except the values of the bending moments at the center of the plate having b/a ratio equal to 1 which have a very big difference in results. The reason behind that high difference is that at b/a ratio equal to 1, the magnitude of the moments should be somewhere around 0 as it seen from the table that the sign of the moments for ratios smaller than 1 turn from –ve to +ve or from +ve to –ve when the ratios become greater than 1. However, both the Ritz and FEM solutions return small value of moments at this point, which results this error. Even though the high difference in the moment value, this difference will not have a big effect on design since the moments at the center (if not zero) are much smaller than the moments at the boundaries.

5.5 Closure

In this chapter, the thesis discussed the analysis of uniformly loaded plates with two opposite sides unsymmetrical using the Galerkin and Ritz methods. Since the boundary conditions in the discussed 9 cases are more complicated than the cases discussed in the previous chapters, these study methods failed to get proper solutions when all the boundary equations were tried to be satisfied. However, when just the essential BCs were applied with the Ritz method, it successfully got out with accurate bending solutions and great results for all the cases. These outcomes are from the provided tables in this chapter that compares the results with the FEM solution derived using COMSOL Multiphysics software and Timoshenko's solution [1] (if available).

CHAPTER 6

UNIFORMLY LOADED RECTANGULAR PLATE WITH CORNER SUPPORTS

6.1 Introduction

This chapter represents the analysis of the last case of the uniformly loaded rectangular plates, which is the FFFF plate case. This plate case is discussed in a separate chapter because it is different than the previous case in terms of boundary equations. The analyzed plate does not have any support at any of the four edges, so it is free at all the edges. In order to make it stable structure, four supports (pins or columns) are places at the four corners of the plate. The geometry and the plate configuration are shown in Figure 13. The free edges are at $x = -a/2$, $x = a/2$, $y = -b/2$ & $y = b/2$, while the supports are at the coordinates $(-a/2, -b/2)$, $(-a/2, b/2)$, $(a/2, -b/2)$ & $(a/2, b/2)$.

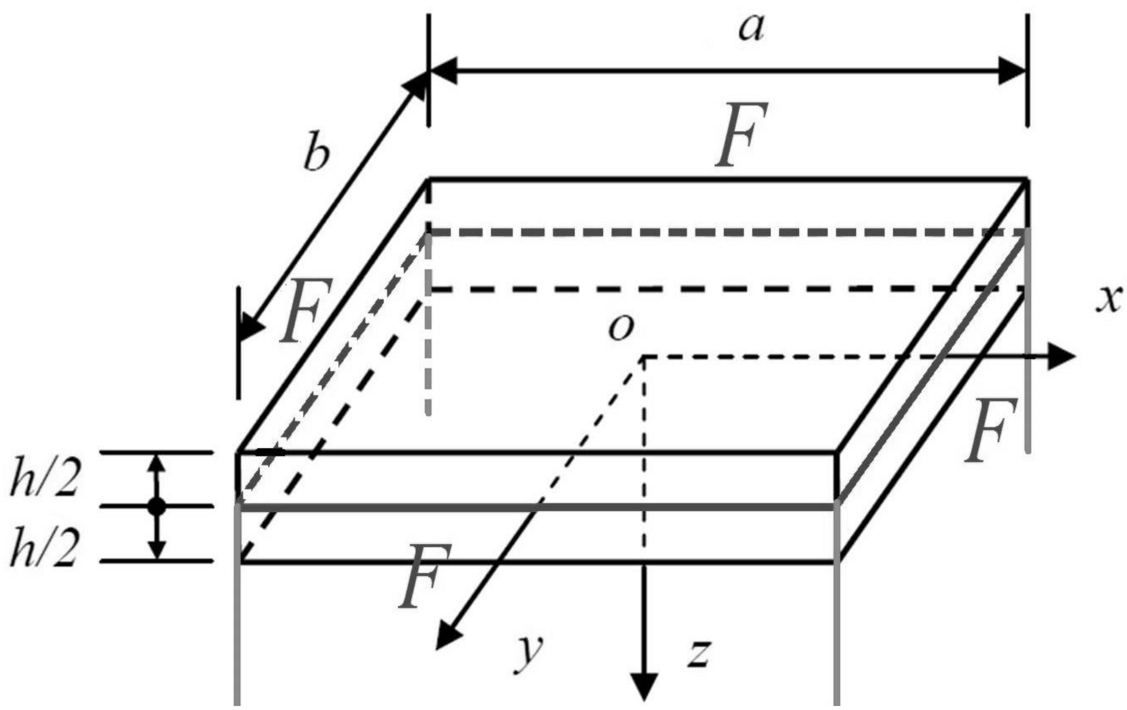

Figure 13 Configuration of Plates with Corner Supports

6.2 Plate Bending Solutions and Comparison of Results

The process of analysis to derive solutions for uniformly loaded rectangular plates with corner supports is similar to the procedure used over the whole research which is discussed with details in sections 2.4 and 2.5. The derived solutions in this chapter are compared with the solution derived by Timoshenko in his book “Theory of Plates and Shells” [1] and the FEM solution derived using COMSOL Multiphysics. As a part of the methodology used in the study, the values of D (the flexural rigidity of the plate), q (uniform load), a (width of the plate) are always used to be 1 since they are used as scaling parameters. As a result of that, the derived solutions for the deflection, moments and shears of the plates are in the form of functions that are polynomials of x & y multiplied by the scaling parameters$a$, q and D raised to some power. The value of b

(length of the plate) is set to be equal to the ratio of b/a and in this chapter, Poisson's Ratio (ν) is set to have the value 0.25. The value of the used ν is different than the rest of the cases because Timoshenko [1] used this value for this case in his book. In order to be able to compare the results, then the same ν should be used.

The previously mentioned configuration of the plate provides a total of 12 boundaries boundary conditions (2 at every edge & 1 at each corner) as follows:

$$(M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0 \ \& \ (V_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + (2-\nu)\frac{\partial^2 w}{\partial x \partial y^2})_{x=-a/2} = 0$$

$$(M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0 \ \& \ (V_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + (2-\nu)\frac{\partial^2 w}{\partial x \partial y^2})_{x=a/2} = 0$$

$$(M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0 \ \& \ (V_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-\nu)\frac{\partial^2 w}{\partial y \partial x^2})_{y=-b/2} = 0$$

$$(M_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \ \& \ (V_y)_{y=b/2} = (\frac{\partial^2 w}{\partial y^2} + (2-\nu)\frac{\partial^2 w}{\partial y \partial x^2})_{y=b/2} = 0$$

$$(w)_{x=-a/2,y=-b/2} = 0 \quad \& \quad (w)_{x=-a/2,y=b/2} = 0$$

$$(w)_{x=a/2,y=-b/2} = 0 \quad \& \quad (w)_{x=a/2,y=b/2} = 0 \qquad (6.1)$$

In this thesis, this case has been solved three times. The first time using the Galerkin method using all the BCs, The second time with Ritz method applying all the 12 BCs and the third time with the Ritz method but by just applying the essential BCs (4 BCs at the corners).

The starting function of deflection in all the solutions was assumed to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} x^{2i} y^{2j} \qquad (6.2)$$

This function does not satisfy any of the BCs by itself but it assure a symmetric solution for the plate around x & y axes. The boundary conditions are satisfied by the help of Mathematica software.

In the study, the first two applied methods found solutions for the case for values of n up to 5, while the third method got out with solutions for values of n up to 6. Since Timoshenko just provided a solution for just b/a ratio equal to 1, the derived solutions were compared with Timoshenko's solution for just 1 b/a ratio in Table 30.

Table 30 Comparison of results for uniformly loaded rectangular FFFF plate (Galerkin – G, Ritz 'satisfying all BCs' – R1, Ritz 'satisfying essential BCs only' – R & Timoshenko – T) for ($\nu = 0.25$), $n = 5$ or 6

$n = 5$						
b/a	**Method**	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{y(\frac{a}{2},0)} * qa^2$
1	G	0.0256337	0.108659	0.108295	0.154055	0.153603
	T	0.02570	0.1109	0.1109	0.1527	0.1527
	% Diff.	0.26%	2.02%	2.35%	-0.89%	-0.59%
$n = 5$						
b/a	**Method**	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{y(\frac{a}{2},0)} * qa^2$
1	R1	0.0256097	0.101819	0.108131	0.149728	0.153536
	T	0.02570	0.1109	0.1109	0.1527	0.1527
	% Diff.	0.35%	8.19%	2.50%	1.95%	-0.55%
$n = 6$						
b/a	**Method**	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{y(\frac{a}{2},0)} * qa^2$
1	R2	0.0256695	0.110752	0.110752	0.152113	0.152113
	T	0.02570	0.1109	0.1109	0.1527	0.1527
	% Diff.	0.12%	0.13%	0.13%	0.38%	0.38%

It is noted from Table 30 that the Galerkin method and the first Ritz method solutions have good agreement with the Timoshenko's solution for $b/a = 1$ but not as good as the results of the second Ritz method solution. Since the previous table provides a comparison for just 1 b/a ratio, a FEM solution using COMSOL software was derived for other b/a ratios and compared with the research solutions in Table 31 & Table 32.

Table 31 Comparison of results for uniformly loaded rectangular FFFF plate (Galerkin – G, Ritz 'satisfying all BCs' – R1 & COMSOL) for (ν=0.25), $n = 5$

$n = 5$						
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{y(\frac{a}{2},0)} * qa^2$
0.5	G	0.0141443	0.122018	0.0204808	0.132127	0.0565915
	COMSOL	0.01430248	0.12221515	0.02017219	0.12985123	0.05579468
	% Diff.	1.11%	0.16%	-1.53%	-1.75%	-1.43%
0.75	G	0.0170659	0.114225	0.053608	0.13842	0.0990427
	COMSOL	0.01720505	0.11727465	0.05641425	0.13868973	0.09702503
	% Diff.	0.81%	2.60%	4.97%	0.19%	-2.08%
1	G	0.0256337	0.108659	0.108295	0.154055	0.153603
	COMSOL	0.02582094	0.11055548	0.11055551	0.15140997	0.15141164
	% Diff.	0.73%	1.72%	2.04%	-1.75%	-1.45%
1.5	G	0.0790538	0.106158	0.273816	0.18231	0.320887
	COMSOL	0.07977526	0.09530157	0.26818398	0.18486228	0.30475949
	% Diff.	0.90%	-11.39%	-2.10%	1.38%	-5.29%

$n = 5$						
b/a	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,\frac{b}{2})} * qa^2$	$M_{y(\frac{a}{2},0)} * qa^2$
0.5	R1	0.0141088	0.128029	0.0216611	0.1367	0.0580035
	COMSOL	0.01430248	0.12221515	0.02017219	0.12985123	0.05579468
	% Diff.	1.35%	-4.76%	-7.38%	-5.27%	-3.96%
0.75	R1	0.0170727	0.114948	0.0550377	0.137353	0.0960924
	COMSOL	0.01720505	0.11727465	0.05641425	0.13868973	0.09702503
	% Diff.	0.77%	1.98%	2.44%	0.96%	0.96%
1	R1	0.0256097	0.101819	0.108131	0.149728	0.153536
	COMSOL	0.02582094	0.11055548	0.11055551	0.15140997	0.15141164
	% Diff.	0.82%	7.90%	2.19%	1.11%	-1.40%
1.5	R1	0.0790297	0.0995365	0.26764	0.18429	0.320908
	COMSOL	0.07977526	0.09530157	0.26818398	0.18486228	0.30475949
	% Diff.	0.93%	-4.44%	0.20%	0.31%	-5.30%

Table 32 Comparison of results for uniformly loaded rectangular FFFF plate (Ritz 'satisfying only essential BCs' – R2 & COMSOL) for (ν=0.25), $n = 6$

$n = 6$						
b/a	**Method**	$w_{(0,0)}$ $*\frac{qa^4}{D}$	$M_{x(0,0)}$ $*qa^2$	$M_{y(0,0)}$ $*qa^2$	$M_{x(0,\frac{b}{2})}$ $*qa^2$	$M_{y(\frac{a}{2},0)}$ $*qa^2$
0.5	R2	0.0142451	0.122283	0.020505	0.130188	0.0561559
	COMSOL	0.01430248	0.12221515	0.02017219	0.12985123	0.05579468
	% Diff.	0.40%	-0.06%	-1.65%	-0.26%	-0.65%
0.75	R2	0.0171041	0.117214	0.0563945	0.139282	0.0976544
	COMSOL	0.01720505	0.11727465	0.05641425	0.13868973	0.09702503
	% Diff.	0.59%	0.05%	0.04%	-0.43%	-0.65%
1	R2	0.0256695	0.110752	0.110752	0.152113	0.152113
	COMSOL	0.02582094	0.11055548	0.11055551	0.15140997	0.15141164
	% Diff.	0.59%	-0.18%	-0.18%	-0.46%	-0.46%
1.5	R2	0.0795123	0.0958313	0.26836	0.185625	0.305278
	COMSOL	0.07977526	0.09530157	0.26818398	0.18486228	0.30475949
	% Diff.	0.33%	-0.56%	-0.07%	-0.41%	-0.17%

Table 31 & Table 32 support the outcomes that has been concluded from Table 30. The first two solutions derived by satisfying all the BCs provide good solutions but not as good or accurate as the third solution that just apply the essential BCs to the Ritz method. It is clear from the last tables that the error for almost all the values is less than 1%.

6.3 Closure

By the end of this chapter, the methods applied in this research approved their ability to analyze any uniformly loaded rectangular plate, especially for the Ritz method. It can be concluded from the previous four chapters that if the Ritz method is not forced to satisfy all the boundary conditions, then it provides better and more accurate solutions.

CHAPTER 7

FULLY SIMPLY SUPPORTED RECTANGULAR PLATES WITH NON-UNIFORMLY DISTRIBUTED LOADINGS

7.1 Introduction

After testing the research methods in deriving solutions for all the possible cases of uniformly loaded rectangular plates, which resulted in great accurate results, the thesis in this chapter tests the possibility of applying the before mentioned methods in the analysis of rectangular plates but with non-uniformly distributed loadings. This part of the research do not include the analysis of all the possible cases, but it studies the use of Galerkin and Ritz methods to analyze two representative cases. The selected cases are for fully simply supported rectangular plates with different loadings. The geometry and the plate configuration of the analyzed plates are shown in Figure 14. The simply supported edges are at $x = -a/2$, $x = a/2$, $y = -b/2$ & $y = b/2$.

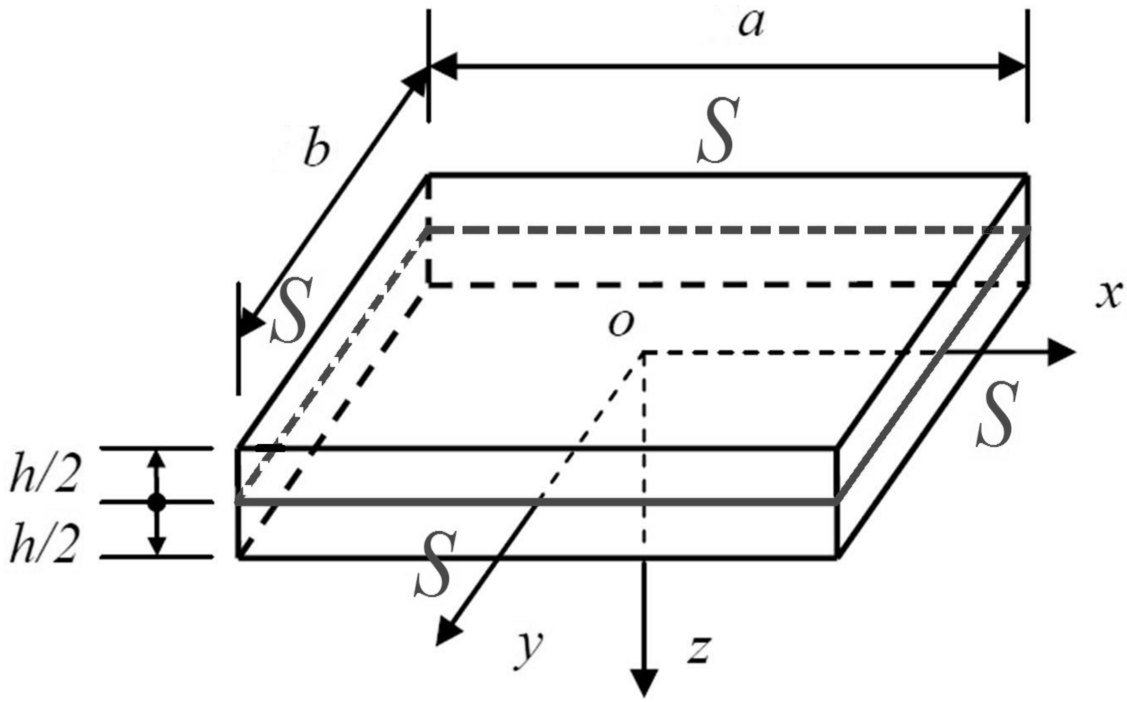

Figure 14 Configuration of Fully Simply Supported Plates

The process of analysis to derive solutions for non-uniformly loaded fully simply supported rectangular plates, is similar to the methodology of analysis used for the same plate with uniformly distributed loading, which is discussed with details in sections 2.4 and 2.5. The only few differences are the equation of the load q, and the starting assumed function of deflection w. The discussed cases in this chapter are discussed also in Timoshenko book, "Theory of Plates and Shells" [1], and have available solutions there. The results derived in this chapter are compared with Timoshenko's solution at the end of each section. As a part of the methodology used in the study, the values of D (the flexural rigidity of the plate), q_0(magnitude of the maximum load), a (width of the plate) are always used to be 1 since they are used as scaling parameters. As a result of that, the derived solutions for the deflection, moments and shears of the plates are in the form of functions that are polynomials of x & y multiplied by the scaling parameters$a$, q_0 and D

raised to some power. The value of b (length of the plate) is set to be equal to the ratio of b/a and Poisson's Ratio (v) is set to have the value 0.3.

The previously mentioned configuration of the plate provides a total of 8 boundaries boundary condition (for both studied cases), which are the same equations given in the first discussed case of this study (uniformly loaded SSSS plate) in section 3.2.1. These equations are:

$$(w)_{x=-a/2} = 0 \qquad \& \qquad (M_x)_{x=-a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=-a/2} = 0$$

$$(w)_{x=a/2} = 0 \qquad \& \qquad (M_x)_{x=a/2} = (\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2})_{x=a/2} = 0$$

$$(w)_{y=-b/2} = 0 \qquad \& \qquad (M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=-b/2} = 0$$

$$(w)_{y=b/2} = 0 \qquad \& \qquad (M_y)_{y=-b/2} = (\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2})_{y=b/2} = 0 \qquad (7.1)$$

7.2 Plate Simply Supported from All the Four Edges Under Hydrostatic Pressure

The first chosen non-uniformly distributed loading case to be analyzed in this chapter is the hydrostatic pressure loading. The load is set to equal to zero at the edge $x = -a/2$ and increasing in a triangular shape until it reaches 1 at the opposite edge, $x = a/2$. So the load function q, is a function of x only and it is constant in the y direction. Figure 15 is given to visualize the load. To get this load distribution, the function of q is set to be:

$$q = q_0 \frac{x + {a}/{2}}{a} \tag{7.2}$$

Where q_0 is set to be equal to 1 in analysis.

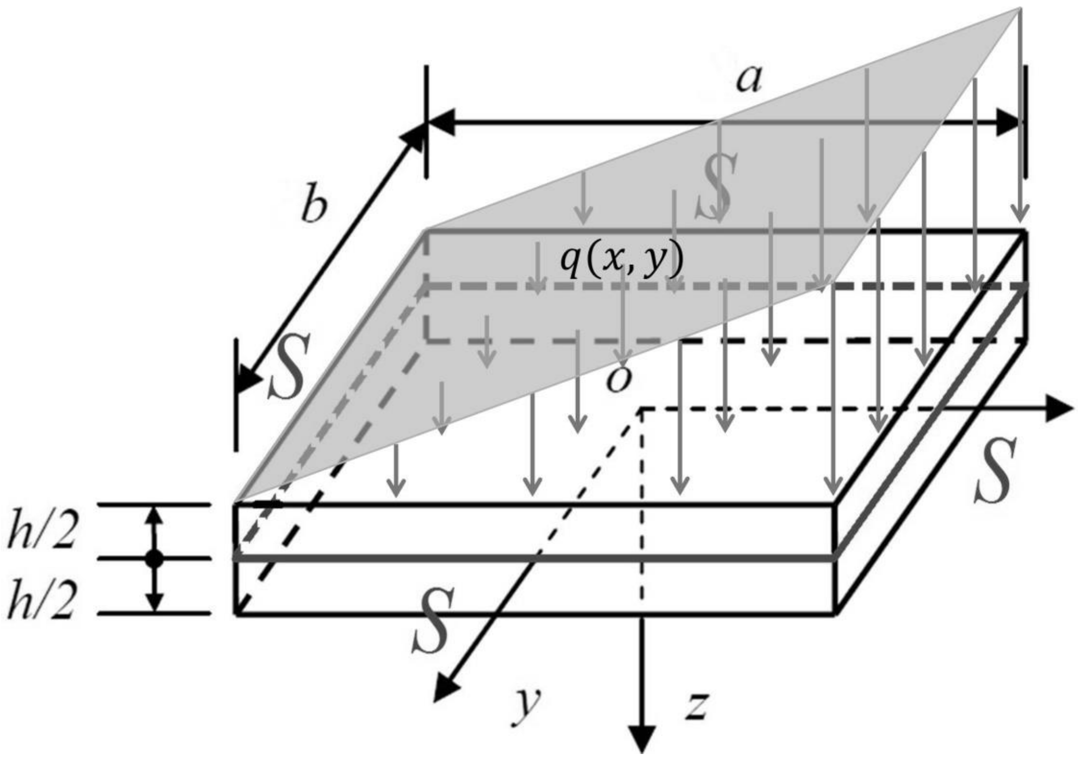

Figure 15 Configuration of Fully Simply Supported Plates Under Hydrostatic Triangular Pressure

In this thesis, this case has been solved two times. The first time using the Galerkin method using all the BCs and another time with Ritz method applying all the BCs also.

The starting function of deflection in the solutions was assumed to be:

$$w = \sum_{i=0}^{2n} \sum_{j=0}^{n} C_{i,j} x^i y^{2j} \tag{7.3}$$

This function does not satisfy any of the BCs by itself as it is clear. However, it assures a symmetric solution for the plate around x axis only. It is true that the boundaries are symmetric in both x and y directions, but the load is symmetric around the x axis only.

Therefore, the deflection should be symmetric around x axis only. The boundary conditions are satisfied by the help of Mathematica software.

In the study, the applied methods found solutions for this case for values of n up to 4, and the results were very similar to the results derived by Timoshenko [1]. For this case specifically, Timoshenko provided a solution table including 30 b/a ratios with 22 different evaluated values including deflections, moment and shears for all the b/a ratios. Since this includes large number of results to be compared with, the derived solutions were compared with just some representative results of Timoshenko. Theses comparisons are given in Table 33 & Table 34.

Table 33 Comparison of results for SSSS plate under hydrostatic pressure (Galerkin – G, & Timoshenko – T) for ($\nu = 0.3$), $n = 4$

$n = 4$						
b/a	Method	$w_{(0,0)}$ $* \frac{q_0 a^4}{D}$	$w_{(\frac{a}{4},0)}$ $* \frac{q_0 a^4}{D}$	$M_{x(-\frac{a}{4},0)}$ $* q_0 a^2$	$M_{x(0,0)}$ $* q_0 a^2$	$M_{x(\frac{a}{4},0)}$ $* q_0 a^2$
1	G	0.00203117	0.00162737	0.0131077	0.0239364	0.0258112
	T	0.00203	0.00162	0.0132	0.024	0.0259
	% Diff.	-0.06%	-0.45%	0.70%	-0.15%	0.34%
1.2	G	0.00282519	0.0022083	0.0178474	0.0313197	0.0317485
	T	0.00282	0.00221	0.0179	0.031	0.0318
	% Diff.	-0.18%	0.08%	0.29%	-0.06%	0.16%
1.5	G	0.00386122	0.00295756	0.0239642	0.0404957	0.0388553
	T	0.00386	0.00296	0.0239	0.041	0.0388
	% Diff.	-0.03%	0.08%	-0.27%	0.26%	-0.14%
1.7	G	0.00441735	0.00335727	0.0272309	0.0452513	0.0424574
	T	0.00441	0.00335	0.0272	0.045	0.0424
	% Diff.	-0.17%	-0.22%	-0.11%	0.33%	-0.14%
2	G	0.00506478	0.00381708	0.0309264	0.0508541	0.046348
	T	0.00506	0.00382	0.0309	0.051	0.0463
	% Diff.	-0.09%	0.08%	-0.09%	-0.11%	-0.10%
5	G	0.00653649	0.00487087	0.0390878	0.0629207	0.0554853
	T	0.00648	0.00483	0.0389	0.062	0.0546
	% Diff.	-0.87%	-0.85%	-0.48%	-1.00%	-1.62%

b/a	Method	$M_{y(0,0)}$ $* q_0a^2$	$M_{y(\frac{a}{4},0)}$ $* q_0a^2$	$R_{(-\frac{a}{2},0)}$ $* q_0a$	$R_{(-\frac{a}{2},\pm\frac{b}{4})}$ $* q_0a$	$R_{(\frac{a}{2},\pm\frac{b}{4})}$ $* q_0a$
1	G	0.023936	0.0207217	0.125237	0.100086	0.257783
	T	0.0239	0.0207	0.126	0.098	0.256
	% Diff.	-0.15%	-0.10%	0.61%	-2.13%	-0.70%
1.2	G	0.0250259	0.0212976	0.140965	0.11742	0.279908
	T	0.025	0.0213	0.144	0.114	0.276
	% Diff.	-0.10%	0.01%	2.11%	-3.00%	-1.42%
1.5	G	0.0248284	0.0210709	0.161956	0.128907	0.296923
	T	0.0249	0.021	0.159	0.132	0.297
	% Diff.	0.29%	-0.34%	-1.86%	2.34%	0.03%
1.7	G	0.0241915	0.0205687	0.17083	0.131576	0.299601
	T	0.0243	0.0205	0.164	0.140	0.306
	% Diff.	0.45%	-0.34%	-4.16%	6.02%	2.09%
2	G	0.0231983	0.0196502	0.162545	0.153957	0.312313
	T	0.0232	0.0197	0.168	0.149	0.316
	% Diff.	0.01%	0.25%	3.25%	-3.33%	1.17%
5	G	0.0195881	0.017258	0.171686	0.165216	0.333753
	T	0.0187	0.0166	0.167	0.167	0.335
	% Diff.	-4.75%	-3.96%	-2.81%	1.07%	0.37%

Table 34 Comparison of results for SSSS plate under hydrostatic pressure (Ritz – R, & Timoshenko – T) for $(\nu = 0.3), n = 4$

$n = 4$						
b/a	Method	$w_{(0,0)}$ $* \frac{q_0a^4}{D}$	$w_{(\frac{a}{4},0)}$ $* \frac{q_0a^4}{D}$	$M_{x(-\frac{a}{4},0)}$ $* q_0a^2$	$M_{x(0,0)}$ $* q_0a^2$	$M_{x(\frac{a}{4},0)}$ $* q_0a^2$
1	R	0.00203112	0.00162741	0.0131155	0.0239273	0.0258223
	T	0.00203	0.00162	0.0132	0.024	0.0259
	% Diff.	-0.06%	-0.46%	0.64%	-0.11%	0.30%
1.2	R	0.00282486	0.00220819	0.0179264	0.0312884	0.0317053
	T	0.00282	0.00221	0.0179	0.031	0.0318
	% Diff.	-0.17%	0.08%	-0.15%	0.04%	0.30%
1.5	R	0.00386113	0.00295739	0.0240302	0.0404803	0.0388345
	T	0.00386	0.00296	0.0239	0.041	0.0388
	% Diff.	-0.03%	0.09%	-0.54%	0.29%	-0.09%
1.7	R	0.00441849	0.00335619	0.0272956	0.0453363	0.0423182
	T	0.00441	0.00335	0.0272	0.045	0.0424
	% Diff.	-0.19%	-0.18%	-0.35%	0.14%	0.19%
2	R	0.00506433	0.00381777	0.030911	0.050805	0.0464463
	T	0.00506	0.00382	0.0309	0.051	0.0463
	% Diff.	-0.09%	0.06%	-0.04%	-0.01%	-0.32%
5	R	0.00653286	0.00487071	0.0392247	0.0627396	0.0554956
	T	0.00648	0.00483	0.0389	0.062	0.0546
	% Diff.	-0.82%	-0.84%	-0.83%	-0.71%	-1.64%

b/a	Method	$M_{y(0,0)}$ $* q_0a^2$	$M_{y(\frac{a}{4},0)}$ $* q_0a^2$	$R_{(-\frac{a}{2},0)}$ $* q_0a$	$R_{(-\frac{a}{2},\pm\frac{b}{4})}$ $* q_0a$	$R_{(\frac{a}{2},\pm\frac{b}{4})}$ $* q_0a$
1	R	0.0239269	0.0207311	0.12489	0.100389	0.25876
	T	0.0239	0.0207	0.126	0.098	0.256
	% Diff.	-0.11%	-0.15%	0.88%	-2.44%	-1.08%
1.2	R	0.0249935	0.0212656	0.133361	0.124507	0.268731
	T	0.025	0.0213	0.144	0.114	0.276
	% Diff.	0.03%	0.16%	7.39%	-9.22%	2.63%
1.5	R	0.0248378	0.0210387	0.15421	0.135575	0.29266
	T	0.0249	0.021	0.159	0.132	0.297
	% Diff.	0.25%	-0.18%	3.01%	-2.71%	1.46%
1.7	R	0.0242687	0.0204776	0.152945	0.150676	0.293288
	T	0.0243	0.0205	0.164	0.140	0.306
	% Diff.	0.13%	0.11%	6.74%	-7.63%	4.15%
2	R	0.0231692	0.0197054	0.167967	0.148308	0.319175
	T	0.0232	0.0197	0.168	0.149	0.316
	% Diff.	0.13%	-0.03%	0.02%	0.46%	-1.00%
5	R	0.0195105	0.01726	0.174377	0.162191	0.320761
	T	0.0187	0.0166	0.167	0.167	0.335
	% Diff.	-4.33%	-3.98%	-4.42%	2.88%	4.25%

It is noted from Table 33 & Table 34 that both solutions have perfect agreement with the Timoshenko's solution over the whole table, even for the reactions or shear at the supports. This gives an indication that the research methods are suitable even in cases that do not have uniformly distributed loads.

7.3 Plate Simply Supported from All the Four Edges Under Triangular Prism Loading

The second loading case analyzed in this chapter is the triangular prism loading. The load is set to be equal to zero at the edge $x = -a/2$ & $x = a/2$ and increasing in a triangular shape toward the y axis until it reaches 1 on top of the y axis. So again, the load function q, is a function of x only and it is constant in the y direction. Figure 16 gives a general view of the plate with the loading on top of it. To get this load distribution, the function of q is set to be:

$$q = q_0(1 - \frac{|x|}{a/2}) \tag{7.4}$$

Where q_0 is set to be equal to 1 in analysis.

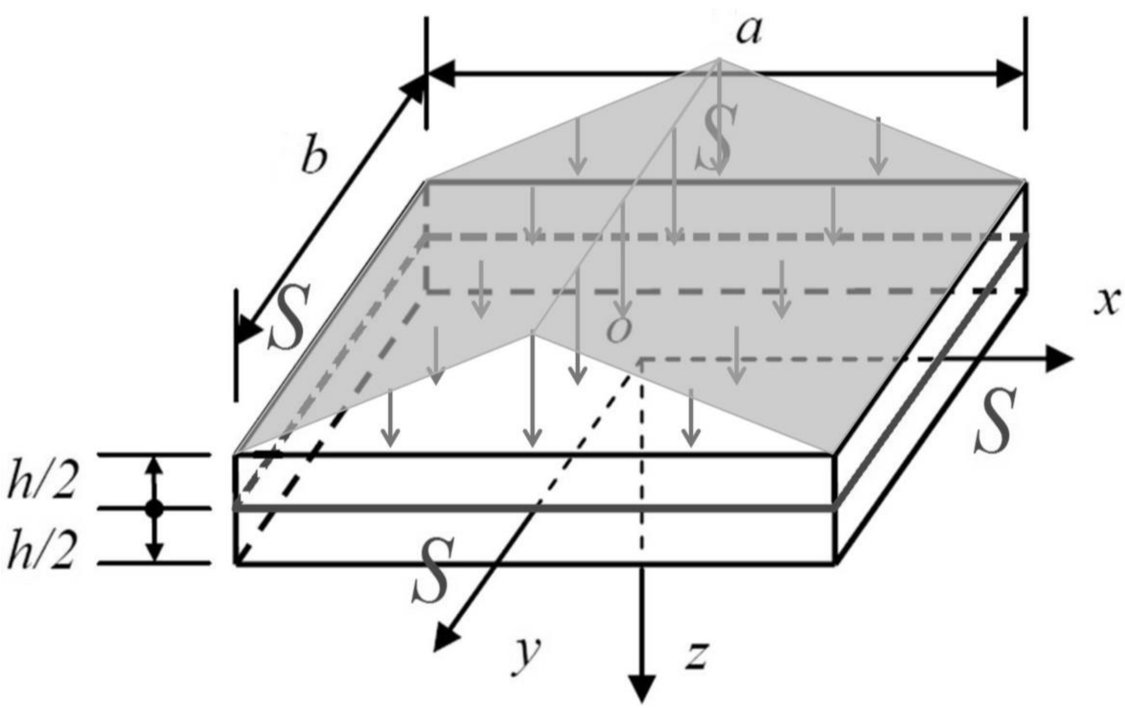

Figure 16 Configuration of Fully Simply Supported Plates Under Triangular Prism Loading

As in the previous case, this case has been solved two times, one time using the Galerkin method applying all the BCs and the second time with the Ritz method applying all the BCs also.

This time, the starting function of deflection in the solutions is a little bit different than the previous case, it was assumed to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} x^{2i} y^{2j} \tag{7.5}$$

The difference is that this function assures a symmetric solution for the plate around x axis as well as the y axis. This is because both, the boundaries and the loading are symmetric in both x and y directions. Therefore, the deflection should be symmetric around x and y axes. The boundary conditions are satisfied by the help of Mathematica software.

As in the previous case, the applied methods found solutions for the case for values of n up to 4, and the results were very similar to the results derived Timoshenko [1]. The comparison between the derived results and the results provided by Timoshenko are shown in Table 35 & Table 36.

Table 35 Comparison of results for SSSS plate under triangular prism loading (Galerkin – G, & Timoshenko – T) for ($\nu = 0.3$), $n = 4$

$n = 4$					
b/a	Method	$w_{(0,0)}$ $* \frac{q_0 a^4}{D}$	$M_{x(0,0)}$ $* q_0 a^2$	$M_{y(0,0)}$ $* q_0 a^2$	$Q_{x(-\frac{a}{2},0)}$ $* q_0 a$
1	G	0.00262702	0.0338953	0.031663	0.153931
	T	0.00263	0.034	0.0317	0.147
	% Diff.	0.11%	0.31%	0.12%	-4.71%
1.2	G	0.00363965	0.0434253	0.033017	0.180006
	T	0.00364	0.0436	0.033	0.173
	% Diff.	0.01%	0.40%	-0.05%	-4.05%
1.5	G	0.00496061	0.0552543	0.032839	0.208562
	T	0.00496	0.0554	0.0329	0.202
	% Diff.	-0.01%	0.26%	0.19%	-3.25%
1.7	G	0.00567101	0.0614495	0.0321155	0.216001
	T	0.00567	0.0615	0.0321	0.214
	% Diff.	-0.02%	0.08%	-0.05%	-0.94%
2	G	0.00649243	0.0683778	0.0306495	0.23529
	T	0.00649	0.0685	0.0306	0.228
	% Diff.	-0.04%	0.18%	-0.16%	-3.20%
3	G	0.0078367	0.0793318	0.0271374	0.251112
	T	0.00783	0.0794	0.027	0.245
	% Diff.	-0.09%	0.09%	-0.51%	-2.49%

b/a	Method	$Q_{y(0,-\frac{b}{2})}$ $* q_0 a$	$V_{x(-\frac{a}{2},0)}$ $* q_0 a$	$V_{y(0,-\frac{b}{2})}$ $* q_0 a$
1	G	0.245656	0.205461	0.309411
	T	0.25	0.199	0.315
	% Diff.	1.74%	-3.25%	1.77%
1.2	G	0.254851	0.228041	0.329479
	T	0.2592	0.222	0.336
	% Diff.	1.68%	-2.72%	1.94%
1.5	G	0.260107	0.247745	0.344476
	T	0.267	0.241	0.354
	% Diff.	2.58%	-2.80%	2.69%
1.7	G	0.262609	0.248198	0.350449
	T	0.2686	0.247	0.360
	% Diff.	2.23%	-0.49%	2.76%
2	G	0.261049	0.259679	0.352029
	T	0.27	0.252	0.366
	% Diff.	3.32%	-3.05%	3.82%
3	G	0.248871	0.259153	0.341205
	T	0.27	0.253	0.366
	% Diff.	7.83%	-2.43%	6.77%

Table 36 Comparison of results for SSSS plate under triangular prism loading (Ritz – R, & Timoshenko – T) for ($\nu = 0.3$), $n = 4$

$n = 4$					
b/a	Method	$w_{(0,0)}$ $* \frac{q_0 a^4}{D}$	$M_{x(0,0)}$ $* q_0 a^2$	$M_{y(0,0)}$ $* q_0 a^2$	$Q_{x(-\frac{a}{2},0)}$ $* q_0 a$
1	R	0.00262702	0.0338973	0.0316626	0.15426
	T	0.00263	0.034	0.0317	0.147
	% Diff.	0.11%	0.30%	0.12%	-4.94%
1.2	R	0.00363788	0.0432361	0.0328652	0.180172
	T	0.00364	0.0436	0.033	0.173
	% Diff.	0.06%	0.83%	0.41%	-4.15%
1.5	R	0.00496041	0.0552336	0.0328229	0.208667
	T	0.00496	0.0554	0.0329	0.202
	% Diff.	-0.01%	0.30%	0.23%	-3.30%
1.7	R	0.00566992	0.06138	0.0320374	0.222523
	T	0.00567	0.0615	0.0321	0.214
	% Diff.	0.00%	0.20%	0.20%	-3.98%
2	R	0.00649213	0.0683422	0.0306236	0.234879
	T	0.00649	0.0685	0.0306	0.228
	% Diff.	-0.03%	0.23%	-0.08%	-3.02%
3	R	0.00783479	0.0792048	0.0270549	0.26053
	T	0.00783	0.0794	0.027	0.245
	% Diff.	-0.06%	0.25%	-0.20%	-6.34%

b/a	Method	$Q_{y(0,-\frac{b}{2})}$ $* q_0 a$	$V_{x(-\frac{a}{2},0)}$ $* q_0 a$	$V_{y(0,-\frac{b}{2})}$ $* q_0 a$
1	R	0.245309	0.205823	0.309011
	T	0.25	0.199	0.315
	% Diff.	1.88%	-3.43%	1.90%
1.2	R	0.257779	0.229456	0.334253
	T	0.2592	0.222	0.336
	% Diff.	0.55%	-3.36%	0.52%
1.5	R	0.259875	0.247954	0.344299
	T	0.267	0.241	0.354
	% Diff.	2.67%	-2.89%	2.74%
1.7	R	0.259394	0.255518	0.346964
	T	0.2686	0.247	0.360
	% Diff.	3.43%	-3.45%	3.73%
2	R	0.259663	0.259307	0.350237
	T	0.27	0.252	0.366
	% Diff.	3.83%	-2.90%	4.31%
3	R	0.245234	0.269018	0.336159
	T	0.27	0.253	0.366
	% Diff.	9.17%	-6.33%	8.15%

Table 35 & Table 36 are showing great agreement with the results of Timoshenko, exactly as noticed from the tables of the previous case.

Note: Deep looking in the tables shows that the values of the results provided by Timoshenko for Q_x & V_x and Q_y & V_y are replaced with each other because they were probably switched by mistake in Timoshenko's book.

7.4 Closure

The overall conclusion by the end of this chapter is that both the Galerkin and the Ritz methods are applicable and can be used to accurately analyze non-uniformly loaded rectangular plates. However, this conclusion cannot be generalized for any case that has these conditions. Further testing and study should be done for these two methods on other cases to go over their limitations.

CHAPTER 8

UNIFORMLY LOADED OTHER PLATES SHAPES

8.1 Introduction

In the previous chapters, the thesis proved the ability of the research methods, especially the Ritz method, to accurately analyze any uniformly loaded rectangular plate. It also showed the possibility of using them to analyze some rectangular plates under non-uniform loads. In this chapter, the study examines the ability of the research methods to analyze uniformly loaded non-rectangular plates. This chapter does not go over all the possible shapes, but it just studies the use of Galerkin and Ritz methods in some representative cases. The selected shapes are the equilateral triangle shape and the elliptical shape.

As in all the previous chapters, the procedure of applying each method continues in the same methodology and this is one of the main advantages of the study methods. Since the boundaries of the plates discussed in this chapter are not always horizontal or vertical as in the rectangular plates, there will be some differences in the way of defining the boundary conditions. This chapter presents the derived Galerkin and Ritz solutions for 5 uniformly loaded plates cases (3 cases of equilateral triangular plates & 2 cases of elliptical plates) and compare these results with the solution derived by Timoshenko in his book “Theory of Plates and Shells” [1] (if available) and the FEM solution derived using the help of COMSOL Multiphysics. As a part of the methodology used in the study, the values of D (the flexural rigidity of the plate), q (uniform load), a (width of the triangular plate & half the width in the case of elliptical plates) since they are used as scaling parameters. As a result of that, the derived solutions for the deflection, moments and shears of the plates are in the form of functions that are polynomials of x & y multiplied by the scaling parameters$a$, q and D raised to some power. The value of b (half the length of the elliptical plate) is set to be equal to the ratio of b/a and in most of the cases in this chapter Poisson's Ratio (ν) is set to have the value 0.3 (unless stated to different with some reasons).

8.2 Equilateral Triangular Plates

In this section, the chapter discusses the analysis of the first non-rectangular plate shape which is the equilateral triangle shape. This shape has been analyzed under uniformly distributed load with three different boundary conditions, which are SSS, CCC & CCS. These boundaries cases have been chosen because Timoshenko [1] has provided some solutions for them in his book. To set the configuration of the edges of the analyzed plates, the length of each edge was set to be equal to $\frac{2a}{\sqrt{3}}$, and the coordinate of the corners of the triangles are $\left(\frac{2a}{3}, 0\right), \left(-\frac{a}{3}, \frac{a}{\sqrt{3}}\right)$ & $(-\frac{a}{3}, -\frac{a}{\sqrt{3}})$ as shown in Figure 17, Figure 18 & Figure 19. This configuration provides the following edges equations, the equation of the vertical edge is $x = -\frac{a}{3}$, the equation of the upper inclined edge is $\frac{x}{\sqrt{3}} - y = \frac{2a}{3\sqrt{3}}$ and at the lower inclined edge, the equation is $\frac{x}{\sqrt{3}} + y = \frac{2a}{3\sqrt{3}}$. Since each edge provides 2 BCs, the total number of BCs available for all the discussed cases is 6.

8.2.1 Equilateral Triangular Plate Simply Supported from All Edges

The first plate case analyzed in this section is the SSS equilateral triangular plate under uniform loading. So the three edges are simply supported as shown in Figure 17.

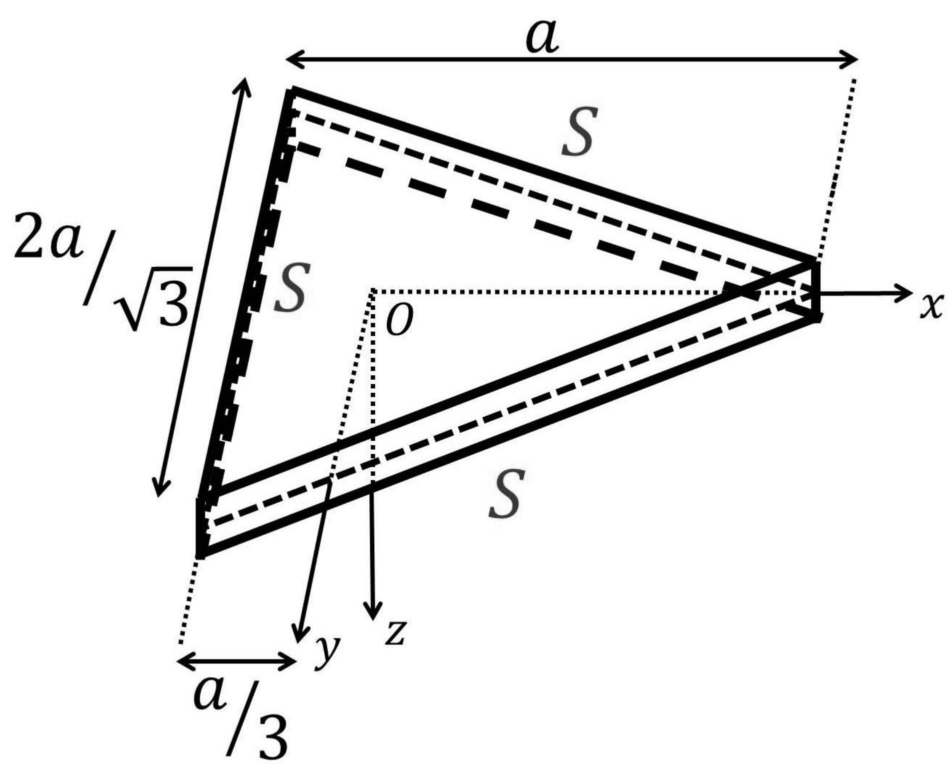

Figure 17 Configuration of Fully Simply Supported Equilateral Triangular Plate

These boundaries provide the following boundary conditions:

$$(w)_{x=-a/3} = 0 \qquad \& \qquad (M_x)_{x=-a/3} = \left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right)_{x=-a/3} = 0$$

$$(w)_{y={}^{2a}/_{3\sqrt{3}}-{}^{x}/_{\sqrt{3}}} = 0 \qquad \& \qquad (M_n)_{y={}^{2a}/_{3\sqrt{3}}-{}^{x}/_{\sqrt{3}}} = 0$$

$$(w)_{y={}^{x}/_{\sqrt{3}}-{}^{2a}/_{3\sqrt{3}}} = 0 \qquad \& \qquad (M_n)_{y={}^{x}/_{\sqrt{3}}-{}^{2a}/_{3\sqrt{3}}} = 0 \qquad (8.1)$$

As it is seen in the boundary conditions equations, there is a new term added here and it is M_n, which is defined to be the bending moment per unit length of a section of the plate perpendicular to the direction normal to the section. In rectangular plates, the normal direction was either in the x or y directions since there was no inclined edges. Therefore, the used equations of moments at the boundaries in rectangular plates was either M_x or M_y (see equations (2.1) & (2.2) in Section 2.2).The equation of M_n is a little bit longer than the equations of M_x and M_y and it is defined as:

$$M_n = -\mathrm{D}\left[\nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + (1-\nu)\left({n_x}^2 \frac{\partial^2 w}{\partial x^2} + {n_y}^2 \frac{\partial^2 w}{\partial y^2} + 2n_x n_y \frac{\partial^2 w}{\partial x \partial y}\right)\right] \quad (8.2)$$

where: n_x is the x component of the unit vector normal to the section and n_y is the y component of the unit vector normal to the section.

If one tried to find n_x and n_y for the inclined edges of the studied plate, the results will be:

- For the upper inclined edge: $n_x = \frac{1}{2}$ & $n_y = \frac{\sqrt{3}}{2}$
- For the bottom inclined edge: $n_x = \frac{1}{2}$ & $n_y = -\frac{\sqrt{3}}{2}$

In this study, this case was solved with the Galerkin method and with the Ritz method by applying all the BCs.

In this case, the starting assumed function of deflection for the solution was in the general form:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{m} C_{i,j} x^i y^{2j} \quad (8.3)$$

This equation does not satisfy any of the BCs and the job is left for Mathematica software to deal with them. However, the equation assures symmetric solution around the x axis.

In this study, solutions of this case for values of n up to 10 and values of m up to 5 were derived. The derived solutions for $n = 10$ & $m = 5$ are compared with Timoshenko's solution [1] in Table 37. Since Timoshenko just provided some values of moments and did not provide any value of deflection, another comparison table was built to compare the results with the solution derived using COMSOL software (Table 38).

Table 37 Comparison of results for uniformly loaded SSS equilateral triangular plate (Galerkin – G, Ritz – R & Timoshenko – T) for ($\nu = 0.3$), $n = 10$ & $m = 5$

$n = 10$ & $m = 5$				
Method	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$M_{x(-0.062a,0)}$ $* qa^2$	$M_{y(0.129a,0)}$ $* qa^2$
G	0.0240741	0.0240741	0.0250196	0.0259275
T	0.02407407	0.02407407	0.025	0.026
% Diff.	0.00%	0.00%	-0.89%	-0.11%
$n = 10$ & $m = 5$				
Method	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$M_{x(-0.062a,0)}$ $* qa^2$	$M_{y(0.129a,0)}$ $* qa^2$
R	0.0240748	0.0240743	0.0250201	0.0259273
T	0.02407407	0.02407407	0.025	0.026
% Diff.	0.00%	0.00%	-0.89%	-0.11%

Table 38 Comparison of results for uniformly loaded SSS equilateral triangular plate (Galerkin – G, Ritz – R & COMSOL) for ($\nu = 0.3$), $n = 10$ & $m = 5$

$n = 10$ & $m = 5$					
Method	$w_{(0,0)}$ $* \frac{qa^4}{D}$	$M_{x(0,0)}$ $* qa^2$	$M_{y(0,0)}$ $* qa^2$	$M_{x(-0.062a,0)}$ $* qa^2$	$M_{y(0.129a,0)}$ $* qa^2$
G	0.00102881	0.0240741	0.0240741	0.0250196	0.0259275
COMSOL	0.00103657	0.02419958	0.0241997	0.02514522	0.02606691
% Diff.	0.75%	0.52%	0.52%	0.50%	0.53%

$\boldsymbol{n = 10}$ & $\boldsymbol{m = 5}$					
Method	$\boldsymbol{w_{(0,0)}}$ $* \boldsymbol{\frac{qa^4}{D}}$	$\boldsymbol{M_{x(0,0)}}$ $* \boldsymbol{qa^2}$	$\boldsymbol{M_{y(0,0)}}$ $* \boldsymbol{qa^2}$	$\boldsymbol{M_{x(-0.062a,0)}}$ $* \boldsymbol{qa^2}$	$\boldsymbol{M_{y(0.129a,0)}}$ $* \boldsymbol{qa^2}$
R	0.00102881	0.0240748	0.0240743	0.0250201	0.0259273
COMSOL	0.00103657	0.02419958	0.0241997	0.02514522	0.02606691
% Diff.	0.75%	0.52%	0.52%	0.50%	0.54%

Table 37 shows excellent agreement between the derived solutions and Timoshenko's solution with % difference less than 1%. It also showed that both methods gave exactly the same results. Moreover, Table 38 shows a perfect agreement of the results with the moments and deflections derived by FEM.

8.2.2 Equilateral Triangular Plate Clamped from All Edges

The second plate case analyzed in this section is the CCC equilateral triangular plate under uniform loading. So the three edges are clamped as shown in Figure 18.

These boundaries provide the following boundary conditions:

$$(w)_{x=-a/3} = 0 \qquad \& \qquad (w_x)_{x=-a/3} = (\frac{\partial w}{\partial x})_{x=-a/3} = 0$$

$$(w)_{y={}^{2a}/_{3\sqrt{3}}-{}^{x}/_{\sqrt{3}}} = 0 \qquad \& \qquad (w_n)_{y={}^{2a}/_{3\sqrt{3}}-{}^{x}/_{\sqrt{3}}} = 0$$

$$(w)_{y={}^{x}/_{\sqrt{3}}-{}^{2a}/_{3\sqrt{3}}} = 0 \qquad \& \qquad (w_n)_{y={}^{x}/_{\sqrt{3}}-{}^{2a}/_{3\sqrt{3}}} = 0 \tag{8.4}$$

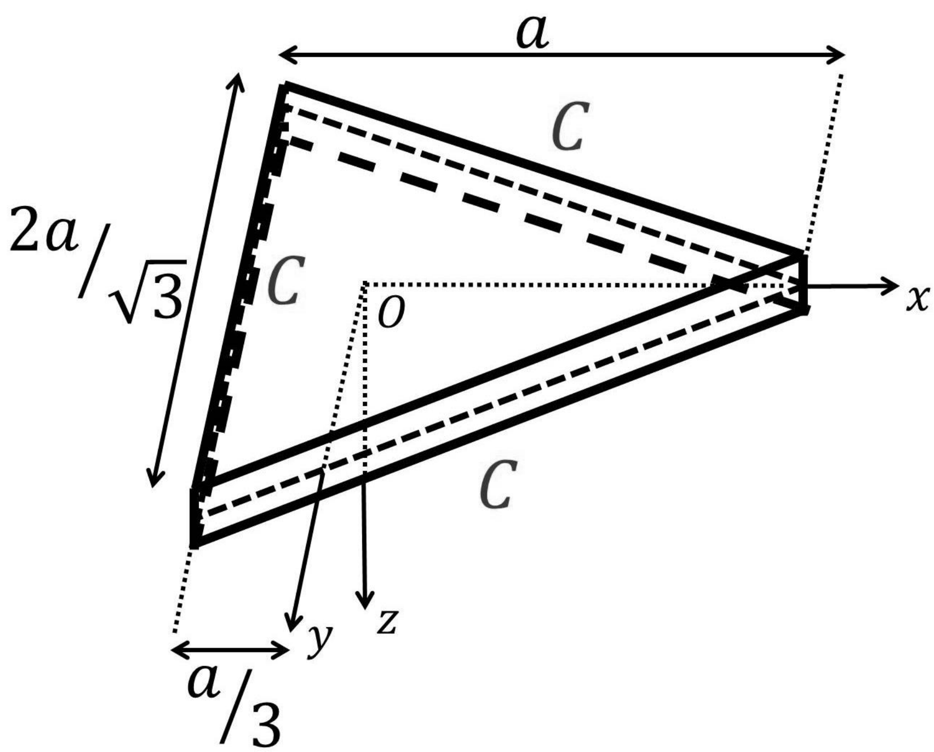

Figure 18 Configuration of Fully Clamped Equilateral Triangular Plate

As it is seen in the boundary conditions equations, there is again a new term given here which is w_n, which is defined to be the slope of a section of the plate perpendicular to the direction normal to the section. In rectangular plates, the normal direction was either in the x or y directions since there was no inclined edges. Therefore, the used equations of slopes at the boundaries in rectangular plates was either w_x or w_y (see equations (2.12) & (2.13) in Section 2.2).The equation of w_n is a little bit longer than the equations of w_x and w_y and it is defined as:

$$w_n = n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y} \qquad (8.5)$$

where: n_x is the x component of the unit vector normal to the section and n_y is the y component of the unit vector normal to the section.

As in the previous case, the magnitudes of n_x and n_y for the inclined edges are:

- For the upper inclined edge: $n_x = \frac{1}{2}$ & $n_y = \frac{\sqrt{3}}{2}$
- For the bottom inclined edge: $n_x = \frac{1}{2}$ & $n_y = -\frac{\sqrt{3}}{2}$

In this study, this case was solved with the Galerkin method and with the Ritz method by applying all the BCs.

The assumed function of deflection is similar to the one used in the previous case, which is:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{m} C_{i,j} x^i y^{2j} \tag{8.6}$$

In this case, the value of v was set to be 0.2 because this is the value used by Timoshenko [1] for this case. Solutions of this case for values of n up to 10 and values of m up to 5 were derived. The derived solutions for $n = 10$ & $m = 5$ are compared with Timoshenko's solution in Table 39. Since there is a big difference in the results and the table did not provide any value of deflection, another comparison table was built to compare the results with the solution derived using COMSOL software (Table 40).

Table 39 shows a high % difference (around 10%) between the derived solutions and Timoshenko's solution. However, both the Galerkin and the Ritz methods gave exactly the same results. To go over the reason behind this difference, another comparison were made with the FEM results derived using COMSOL Multiphysics software, Table 40 shows a perfect agreement of the derived results with the moments and deflections derived by FEM. This means that the derived solutions using the Galerkin and the Ritz methods are more accurate than the solution derived by Timoshenko, this is due to the crude approximations used by Timoshenko since he used the finite difference method in the derivation of his solution.

Table 39 Comparison of results for uniformly loaded CCC equilateral triangular plate (Galerkin – G, Ritz – R & Timoshenko – T) for ($\nu = 0.2$), $n = 10$ & $m = 5$

$n = 10$ & $m = 5$					
Method	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(-\frac{a}{3},0)} * qa^2$	$M_{n(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{n(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$
G	0.0100158	0.0100615	-0.0260857	-0.0263803	-0.0263803
T	0.01100	0.0113	-0.0238	-0.0238	-0.0238
% Diff.	8.95%	10.96%	-9.60%	-10.84%	-10.84%
$n = 10$ & $m = 5$					
Method	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(-\frac{a}{3},0)} * qa^2$	$M_{n(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{n(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$
R	0.0100656	0.0100675	-0.0263185	-0.0263225	-0.0263225
T	0.01100	0.0113	-0.0238	-0.0238	-0.0238
% Diff.	8.49%	10.91%	-10.58%	-10.60%	-10.60%

Table 40 Comparison of results for uniformly loaded CCC equilateral triangular plate (Galerkin – G, Ritz – R & COMSOL) for ($\nu = 0.2$), $n = 10$ & $m = 5$

$n = 10$ & $m = 5$					
Method	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(-\frac{a}{3},0)} * qa^2$	$M_{y(-\frac{a}{3},0)} * qa^2$	$w_{(0,0)} * \frac{qa^4}{D}$
G	0.0100158	0.0100615	-0.0260857	-0.0052172	0.0002979
COMSOL	0.01007237	0.01007136	-0.0267081	-0.0053416	0.00029933
% Diff.	0.56%	0.10%	2.33%	2.33%	0.48%
Method		$M_{x(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{y(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{x(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$	$M_{y(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$
G		-0.0105521	-0.0211043	-0.0105521	-0.0211043
COMSOL		-0.0106833	-0.0213665	-0.0106832	-0.0213665
% Diff.		1.23%	1.23%	1.23%	1.23%

$n = 10$ & $m = 5$					
Method	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(-\frac{a}{3},0)} * qa^2$	$M_{y(-\frac{a}{3},0)} * qa^2$	$w_{(0,0)} * \frac{qa^4}{D}$
R	0.0100656	0.0100675	-0.0263185	-0.0052637	0.00029809
COMSOL	0.01007237	0.01007136	-0.0267081	-0.0053416	0.00029933
% Diff.	0.07%	0.04%	1.46%	1.46%	0.41%
Method		$M_{x(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{y(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{x(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$	$M_{y(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$
R		-0.010529	-0.021058	-0.010529	-0.021058
COMSOL		-0.0106833	-0.0213665	-0.0106832	-0.0213665
% Diff.		1.44%	1.44%	1.44%	1.44%

8.2.3 Equilateral Triangular Plate Clamped from Two Edges and Simply Supported from the Third Edge

The third and the last plate case analyzed in this section is the CCS equilateral triangular plate under uniform loading. The two inclined edges are set to be clamped while the third vertical edge is simply supported as shown in Figure 19.

These boundaries provide the following boundary conditions:

$$(w)_{x=-a/3} = 0 \qquad \& \qquad (M_x)_{x=-a/3} = \left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right)_{x=-a/3} = 0$$

$$(w)_{y={}^{2a}/_{3\sqrt{3}}-{}^{x}/_{\sqrt{3}}} = 0 \qquad \& \qquad (w_n)_{y={}^{2a}/_{3\sqrt{3}}-{}^{x}/_{\sqrt{3}}} = 0$$

$$(w)_{y={}^{x}/_{\sqrt{3}}-{}^{2a}/_{3\sqrt{3}}} = 0 \qquad \& \qquad (w_n)_{y={}^{x}/_{\sqrt{3}}-{}^{2a}/_{3\sqrt{3}}} = 0 \tag{8.7}$$

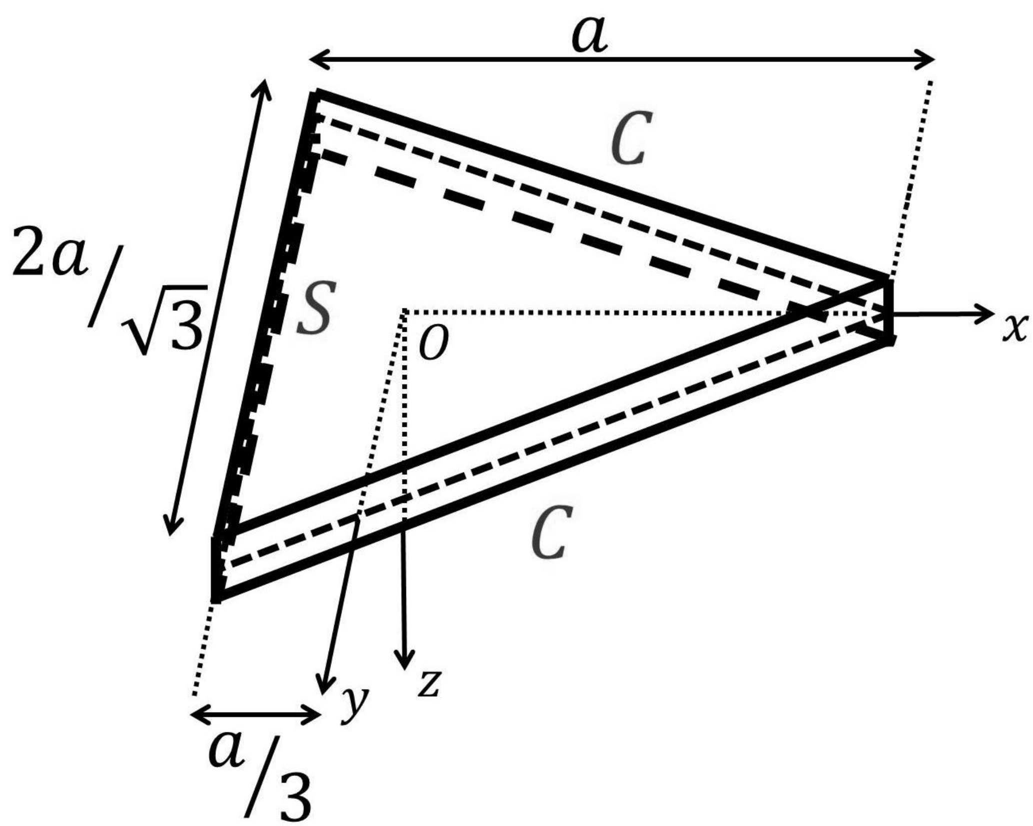

Figure 19 Configuration of Equilateral Triangular Plate Clamped from Two Edges and Simply Supported at the Third Edge

The equation of w_n was defined previously in equation (8.5), and as in the previous two cases, the magnitudes of n_x and n_y for the inclined edges are:

- For the upper inclined edge: $n_x = \frac{1}{2}$ & $n_y = \frac{\sqrt{3}}{2}$
- For the bottom inclined edge: $n_x = \frac{1}{2}$ & $n_y = -\frac{\sqrt{3}}{2}$

In this study, this case was solved again one time with the Galerkin method and another time with the Ritz method by applying all the BCs.

The assumed function of deflection is similar to the one used in the previous two cases, which is:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{m} C_{i,j} x^i y^{2j} \quad (8.8)$$

As in the previous case, the value of v was set to be 0.2 because this is the value used by Timoshenko [1] for this case. Solutions of this case for values of n up to 10 and values of m up to 5 were derived. The derived solutions for $n = 10$ & $m = 5$ are compared with Timoshenko's solution in Table 41. Since there is a big difference in the results and the table did not provide any value of deflection, another comparison table was built to compare the results with the solution derived using COMSOL software (Table 42).

Table 41 Comparison of results for uniformly loaded CCS equilateral triangular plate (Galerkin – G, Ritz – R & Timoshenko – T) for ($\nu = 0.2$), $n = 10$ & $m = 5$

$n = 10$ & $m = 5$					
Method	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(-\frac{a}{3},0)} * qa^2$	$M_{n(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{n(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$
G	0.0109052	0.0133257	0	-0.0317049	-0.0317049
T	0.01470	0.0126	0	-0.0285	-0.0285
% Diff.	25.81%	-5.76%	0.00%	-11.25%	-11.25%
$n = 10$ & $m = 5$					
Method	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(-\frac{a}{3},0)} * qa^2$	$M_{n(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{n(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$
R	0.0108958	0.0133224	0	-0.0316855	-0.0316855
T	0.01470	0.0126	0	-0.0285	-0.0285
% Diff.	25.88%	-5.73%	0.00%	-11.18%	-11.18%

The conclusions from the Table 41 & Table 42 are exactly like the conclusions extracted from Table 39 & Table 40 in the previous case. Table 41 shows a high % difference (around 15% in average) between the derived solutions and Timoshenko's solution. However, both the Galerkin and the Ritz methods gave exactly the same results. This high difference is due to the crude approximations in the results provided by Timoshenko since he used the finite difference method in deriving the solution. Another comparison were made with the FEM results derived using COMSOL Multiphysics software in Table 42, this comparison shows a perfect agreement of the derived results with the moments and deflections derived by FEM.

Table 42 Comparison of results for uniformly loaded CCS equilateral triangular plate (Galerkin – G, Ritz – R & COMSOL) for ($\nu = 0.2$), $n = 10$ & $m = 5$

$n = 10$ & $m = 5$				
Method	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$
G	0.0109052	0.0133257	0.00041922	-0.012682
COMSOL	0.01092466	0.01334955	0.00042138	-0.0129025
% Diff.	0.18%	0.18%	0.51%	1.71%
Method		$M_{y(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{x(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$	$M_{y(\frac{a}{6}-,\frac{a}{2\sqrt{3}})} * qa^2$
G		-0.0253639	-0.012682	-0.0253639
COMSOL		-0.0256554	-0.0129025	-0.0256554
% Diff.		1.14%	1.71%	1.14%
$n = 10$ & $m = 5$				
Method	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$
R	0.0108958	0.0133224	0.00041919	-0.0126742
COMSOL	0.01092466	0.01334955	0.00042138	-0.0129025
% Diff.	0.26%	0.20%	0.52%	1.77%
Method		$M_{y(\frac{a}{6},\frac{a}{2\sqrt{3}})} * qa^2$	$M_{x(\frac{a}{6},-\frac{a}{2\sqrt{3}})} * qa^2$	$M_{y(\frac{a}{6}-,\frac{a}{2\sqrt{3}})} * qa^2$
R		-0.0253484	-0.0126742	-0.0253484
COMSOL		-0.0256554	-0.0129025	-0.0256554
% Diff.		1.20%	1.77%	1.20%

8.3 Elliptical Plates

In this section, the chapter discusses the analysis of another non-rectangular plate shape which is the elliptical shape. This shape has been analyzed under uniformly distributed load with two different boundary conditions, one time when it is fully clamped along its boundary and another time when it is simply supported along its boundary. These boundaries cases have been chosen because Timoshenko [1] has provided some solutions for them in his book. The elliptical plate is set to have its center at the origin $(0,0)$. The width of the plate is set to be $2a$ while the length of the plate is set to be $2b$ as shown in Figure 20 & Figure 21. This configuration of the plate sets the boundary of the plate to follow the general equation of ellipse which is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. Since the whole boundary is defined with one equation, and the whole boundary has the same type of support, then each case will have only two boundary conditions.

8.3.1 Elliptical Plate Clamped from All Edges

This section starts with the analysis of the fully clamped elliptical plate under uniform loading, which means that the whole boundary is clamped as shown in Figure 20.

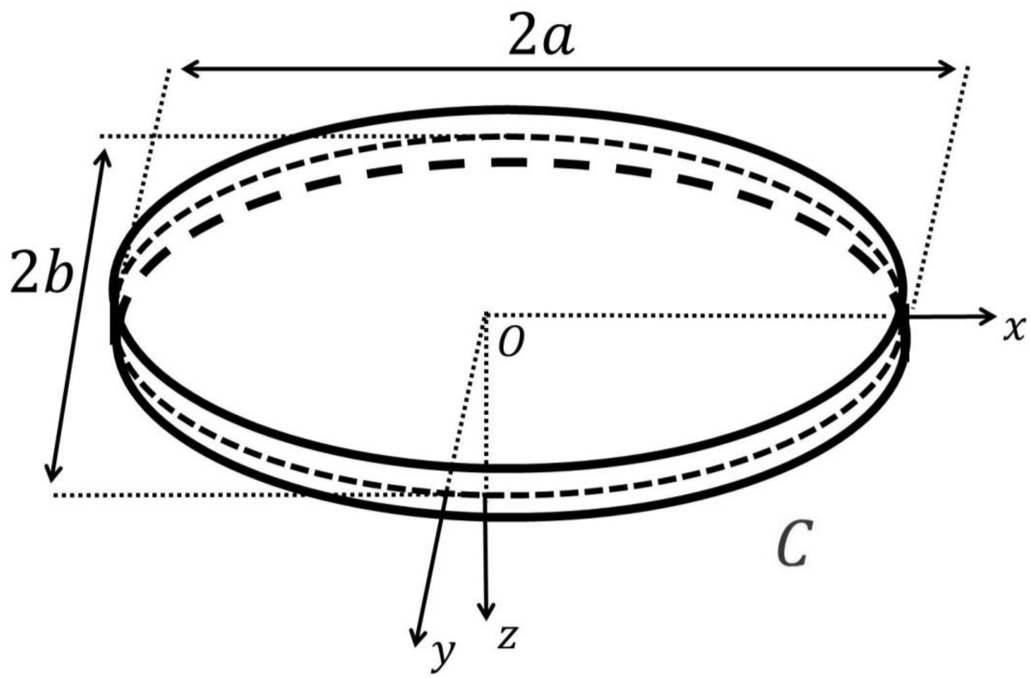

Figure 20 Configuration of Fully Clamped Elliptical Plate

This boundary provides the following boundary conditions:

$$(w)_{y=\mp\frac{b\sqrt{a^2-x^2}}{a}} = 0 \qquad \& \qquad (w_n)_{y=\mp\frac{b\sqrt{a^2-x^2}}{a}} = 0 \tag{8.9}$$

The equation of w_n was previously defined in this chapter (see equation (8.5)). If one tried to find n_x and n_y for the boundary of the plate, the results will be:

- $n_x = \frac{a^2 y}{\sqrt{a^4 y^2 + b^4 x^2}}$
- $n_y = \frac{b^2 x}{\sqrt{a^4 y^2 + b^4 x^2}}$

The uniformly loaded fully clamped elliptical plate is one of the few plate cases that has known exact solution that satisfies exactly both the boundary conditions and the general differential equation of the plates, which make it the easiest plate case to be analyzed.

The exact deflection equation of such a plate is equal to a constant multiplied by the square of the equation of the ellipse. So the equation becomes:

$$w = c\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]^2 \tag{8.10}$$

and this is the equation used by Timoshenko in his book [1]. Therefore, the solution provided by Timoshenko is exact.

In this study, this case was solved with the Galerkin method and with the Ritz method by applying all the BCs. In both solutions, the starting assumed function of deflection for the solution was in the general form:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j} x^{2i} y^{2j} \tag{8.11}$$

This equation does not satisfy any of the BCs but it assures symmetric solution around the x and y axes.

This case is a real test for the accuracy of the results than can be derived using the research method. Looking at equation (8.10) we see that the power of x and y is 4 in this equation. In equation (8.11), if $n = 2$, then the power of x and y become 4.

Therefore, in this study, solutions of this case for values of n up to 4 were derived and compared with the exact Timoshenko's solution [1]. The comparison with the derived solutions for $n = 2$ is given in Table 43.

Table 43 Comparison of results for uniformly loaded fully clamped elliptical plate (Galerkin – G, Ritz – R & Timoshenko – T) for ($\nu = 0.3$), $n = 2$

$n = 2$								
a/b	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,b)} * qa^2$	$M_{y(0,b)} * qa^2$	$M_{x(a,0)} * qa^2$	$M_{y(a,0)} * qa^2$
1	G	0.015625	0.08125	0.08125	-0.0375	-0.125	-0.125	-0.0375
	T	0.01563	0.08125	0.08125	-0.0375	-0.125	-0.125	-0.038
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1.2	G	0.0214201	0.0852043	0.10353	-0.0514082	-0.171361	-0.119	-0.0357001
	T	0.02142	0.08520428	0.10353035	-0.0514082	-0.1713606	-0.119	-0.036
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1.5	G	0.0278926	0.0830579	0.126446	-0.0669421	-0.22314	-0.0991736	-0.0297521
	T	0.02789	0.08305785	0.12644628	-0.0669421	-0.2231405	-0.099	-0.030
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
2	G	0.0338983	0.0745763	0.145763	-0.0813559	-0.271186	-0.0677966	-0.020339
	T	0.03389831	0.07457627	0.14576271	-0.0813559	-0.2711864	-0.068	-0.020
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
5	G	0.0405213	0.0551089	0.16403	-0.097251	-0.32417	-0.0129668	-0.00389
	T	0.04052127	0.05510892	0.16403008	-0.097251	-0.3241701	-0.013	-0.004
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
$n = 2$								
a/b	Method	$w_{(0,0)} * \frac{qa^4}{D}$	$M_{x(0,0)} * qa^2$	$M_{y(0,0)} * qa^2$	$M_{x(0,b)} * qa^2$	$M_{y(0,b)} * qa^2$	$M_{x(a,0)} * qa^2$	$M_{y(a,0)} * qa^2$
1	R	0.015625	0.08125	0.08125	-0.0375	-0.125	-0.125	-0.0375
	T	0.01563	0.08125	0.08125	-0.0375	-0.125	-0.125	-0.038
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1.2	R	0.0214201	0.0852043	0.10353	-0.0514082	-0.171361	-0.119	-0.0357001
	T	0.02142	0.08520428	0.10353035	-0.0514082	-0.1713606	-0.119	-0.036
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1.5	R	0.0278926	0.0830579	0.126446	-0.0669421	-0.22314	-0.0991736	-0.0297521
	T	0.02789	0.08305785	0.12644628	-0.0669421	-0.2231405	-0.099	-0.030
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
2	R	0.0338983	0.0745763	0.145763	-0.0813559	-0.271186	-0.0677966	-0.020339
	T	0.03390	0.07457627	0.14576271	-0.0813559	-0.2711864	-0.068	-0.020
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
5	R	0.0405213	0.0551089	0.16403	-0.097251	-0.32417	-0.0129668	-0.00389
	T	0.04052	0.05510892	0.16403008	-0.097251	-0.3241701	-0.013	-0.004
	% Diff.	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

It is clearly noted from Table 43 that the difference between the results of Timoshenko, Galerkin method and the Ritz method is 0%, which means that both method provided the exact solution automatically by satisfying both the boundary conditions as well as the general differential equation. More comparison tables have been made to compare the results of the Galerkin and the Ritz method when $n = 3$ & $n = 4$ with Timoshenko's and they gave exactly the same results, which means that the coefficients accompanied with terms having power of x and y larger than 4 are having a value of zero. This plate solution in a great indication of how powerful the applied methods are.

8.3.2 Elliptical Plate Simply Supported from All Edges

The last plate case analyzed in this thesis is the fully simply supported elliptical plate under uniform loading, which is simply supported along the whole boundary of the plate as shown in Figure 21.

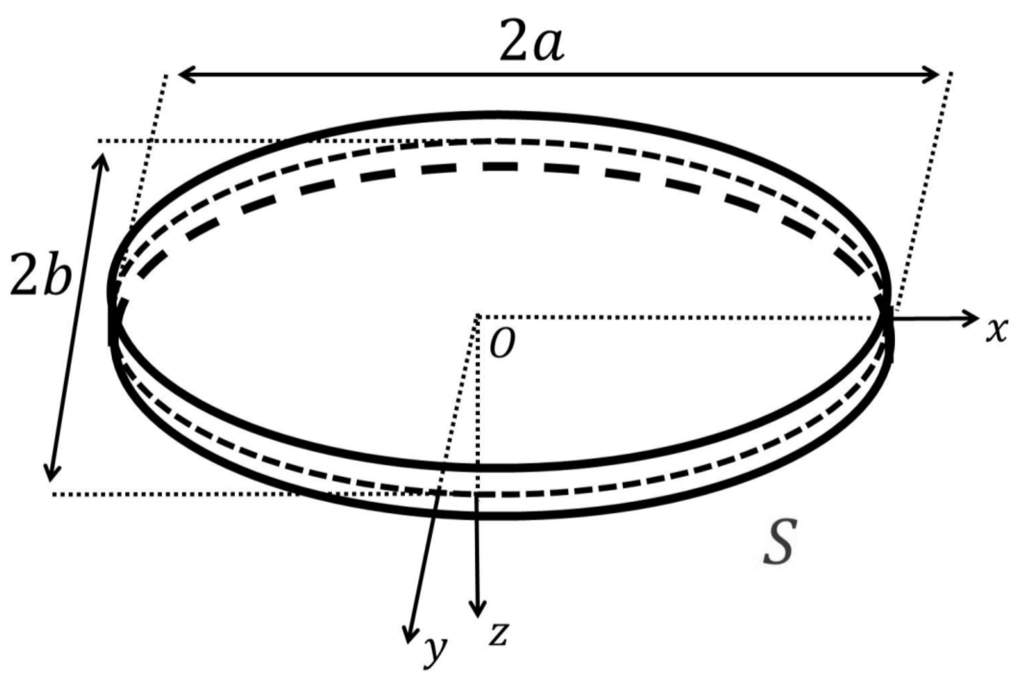

Figure 21 Configuration of Fully Simply Supported Elliptical Plate

This boundary provides the following boundary conditions:

$$(w)_{y=\mp\frac{b\sqrt{a^2-x^2}}{a}} = 0 \qquad \& \qquad (M_n)_{y=\mp\frac{b\sqrt{a^2-x^2}}{a}} = 0 \tag{8.12}$$

The equation of M_n was previously defined in this chapter (see equation (8.2)), and as in the previous case, the values of n_x and n_y are:

- $n_x = \frac{a^2 y}{\sqrt{a^4y^2+b^4x^2}}$
- $n_y = \frac{b^2 x}{\sqrt{a^4y^2+b^4x^2}}$

Since the boundary for this case is simply supported, then it seems that this case is simpler than the case of fully clamped elliptical plate. However, the analysis showed the opposite of that and it appears that this case is one of the most difficult cases to be analyzed. Therefore, the Galerkin method was not able to get out with a proper solution for this plate case. This case has been solved only one time with the Ritz method by just satisfying the essential BC which is the deflection equal to 0 at the boundary. In this solution, the starting assumed function of deflection for the solution was selected to be:

$$w = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{i,j}(1 - (\frac{x}{a})^2 - (\frac{y}{b})^2)x^{2i}y^{2j} \tag{8.13}$$

This equation assures the satisfaction of the essential BC by multiplying the general polynomial equation by the equation of the ellipse which assures that the deflection equal to 0 at the boundaries. The assumed equation also assure symmetric solution around the x and y axes.

In this study, solutions of this case with the Ritz method for values of n up to 3 were derived. The derived solution for $n = 3$ are compared with Timoshenko's solution [1] in Table 44.

Table 44 Comparison of results for uniformly loaded fully simply supported elliptical plate (Ritz – R & Timoshenko – T) for ($\nu = 0.3$), $n = 3$

n=3				
a/b	**Method**	$w_{(0,0)}$ $*\frac{qa^4}{D}$	$M_{x(0,0)}$ $*qa^2$	$M_{y(0,0)}$ $*qa^2$
1	R	0.0637019	0.20625	0.20625
	T	0.06410	0.206	0.206
	% Diff.	0.63%	-0.12%	-0.12%
1.1	R	0.0760637	0.214248	0.235671
	T	0.07601	0.215	0.235
	% Diff.	-0.07%	0.35%	-0.29%
1.2	R	0.0874336	0.218412	0.261699
	T	0.08791	0.219	0.261
	% Diff.	0.54%	0.27%	-0.27%
1.3	R	0.0976333	0.219809	0.284262
	T	0.09799	0.223	0.303
	% Diff.	0.36%	1.43%	6.18%
1.4	R	0.106654	0.219362	0.303636
	T	0.10714	0.223	0.303
	% Diff.	0.46%	1.63%	-0.21%
1.5	R	0.114583	0.217765	0.320232
	T	0.11538	0.222	0.321
	% Diff.	0.69%	1.91%	0.24%
2	R	0.142334	0.204826	0.375585
	T	0.14469	0.21	0.379
	% Diff.	1.63%	2.46%	0.90%
3	R	0.170151	0.184666	0.428495
	T	0.17216	0.188	0.433
	% Diff.	1.17%	1.77%	1.04%
4	R	0.184003	0.173089	0.454546
	T	0.18498	0.18400	0.46500
	% Diff.	0.53%	5.93%	2.25%
5	R	0.191725	0.166206	0.469056
	T	0.19231	0.17	0.48
	% Diff.	0.30%	2.23%	2.28%

Table 44 shows a perfect agreement between the results derived with the Ritz method and the results given by Timoshenko.

8.4 Closure

In this chapter, the thesis proved the ability of applying the Galerkin and Ritz methods in the analysis of uniformly loaded non-rectangular plates to get accurate results. However, it cannot be said that they are valid for any non-rectangular shape until more studies done on it. Further examination of the limitations of the used methods need to be done.

CHAPTER 9

CONCLUSIONS AND RECOMMENDATIONS

9.1 Conclusions

This thesis has presented a great development in the symbolic analysis of thin plates. This research overcomes one of the big challenges of obtaining simple yet accurate polynomial solutions of plate bending with all possible boundary conditions. The obtained solutions are more practical than the existing trigonometric/hyper-trigonometric series solutions, which are complicated and limited to certain boundary conditions. The procedure used in deriving the polynomial solutions is based on two energy-based methods, namely the Galerkin-based weighted residual method and the Ritz method. Based on the results obtained, the following conclusions can be drawn:

1. The proposed energy-based polynomial solution offers an excellent alternative analytical solution to plate bending with different shapes/loads and mixed boundary conditions.
2. The use of Mathematica, symbolic software, greatly reduced the effort of generating polynomials capable of satisfying boundary conditions.
3. Since the solutions are in the form of polynomials, results can be obtained to any desired degree of accuracy by increasing the number of terms and the power of x and y. Another advantage of the methods of research is that the derivation procedure is identical for any plate case.

4. The derived polynomial solutions in this thesis have been presented and showed excellent agreement with the previously established analytical solutions and the FEM solutions.
5. For plates involving simple and clamped edges only, both and Ritz method yield the same level of solution accuracy. Galerkin method does not have the power to deal with cases involving free edges since this case of boundary condition puts a restriction on the method.
6. Ritz method does not require the satisfaction of secondary boundary conditions (shear and moment), which makes it more efficient for handling plates with free edges.
7. Some crude approximations have been discovered in the “Theory of Plates and Shells” book of Timoshenko [1]. With the exception of the error in SSFF case, which is likely to be typo case, the other errors are due to the use of FDM.

9.2 Recommendations

Some suggestions for future research on the application of the presented energy-based methods are given below:

- The proposed methods can be easily extended to solve plates undergoing large deflection. The procedure will produce nonlinear equations to be solved for the polynomial coefficients.

- The proposed methods can be extended to polygon and sector plates for which the analytical solutions are either complicated or not available. If polynomials are not possible, one can try other forms such as trigonometric functions.
- The accuracy of analysis and computations depends on the power and performance of the used PC to handle symbolic computations. The same research could be repeated on a high performance PC capable of performing the symbolic computations. This will enable generating higher order polynomials and hence more accurate solutions.
- It should be noted that the Galerkin and the Ritz methods applied in this research are not limited to plate deflection problems. If the problem can be modeled as a differential equation (or a system of differential equations) with a defined set of boundary conditions, the same procedure can be used to generate the corresponding polynomial solutions. Therefore, the proposed methods can be used to solve problems appearing in other fields of structural engineering or even in the general area of science and engineering.

Appendix A

Following, is the general form of Mathematica code used in this research to derive rectangular plate deflection solutions based on the Galerkin method.

```
Clear["Global`*"]
α={0.5,0.75,1,1.25,1.5,2};q=1;DD=1;a=1;
(*where α is the set of ratios b/a being studied, q is the
load function in x & y, a is the length of the side
parallel to the x axis and DD is the magnitude of the
flexural rigidity of the plate (D)*)

Do[Do[b=α[[k]];
(*where b is the length of the side parallel to the y
axis*)
   w= Sum[c[i,j]   x^i  y^j ,{i,0,n},{j,0,n}];
(*where here ϕ[i,j] is assumed to be  x^ i  y^ j,  w is the
assumed deflection equation and n sets the number of terms
used in the w function*)

(*Defining all the derivations of w*)
wx=D[w,x];wy=D[w,y];wxx=D[w,{x,2}];wyy=D[w,{y,2}];wxy=D[D[w
,x],y];wxxx=D[w,{x,3}];wyyy=D[w,{y,3}];wxxy=D[D[w,{x,2}],y]
;wyyx=D[D[w,{y,2}],x];mx=-DD(wxx+ν        wyy);my=-DD(wyy+ν
wxx);vy=-DD(wyyy+(2-ν) wxxy);vx=-DD(wxxx+(2-ν) wyyx);

(*Listing all the boundary conditions (depending on the
plate boundaries)*)
   eq1=w==0/.x→-a/2;
   eq2=mx==0/.x→-a/2;
   eq3=w==0/.x→a/2;
```

```
eq4=mx==0/.x→ a/2;
eq5=my==0/.y→-b/2;
eq6=vy==0/.y→-b/2;
eq7=my==0/.y→ b/2;
eq8=vy==0/.y→ b/2;

(*Applying SolveAlways function to make ϕ[i,j] satisfying all the boundary conditions*)
eqs={eq1,eq2,eq3,eq4,eq5,eq6,eq7,eq8};
sol=SolveAlways[eqs,{x,y}];
w=Simplify[w/.sol[[1]]];

(*Applying the Galerkin MWR method to solve for parameters c*)
coef=Cases[Cases[Variables[w],Except[x]],Except[y]];
nn=Length[coef];
w=Simplify[w/.Table[coef[[i]]→ C[i],{i,1,nn}]];
Do[P[i]=Coefficient[w,C[i]],{i,1,nn}];
Do[gde[i]=Simplify[D[P[i],{x,4}]+2
D[D[P[i],{x,2}],{y,2}]+D[P[i],{y,4}]],{i,1,nn}];
K=Table[0,{i,1,nn},{j,1,nn}];
F=Table[0,{i,1,nn}];
Do[K[[i,j]]=NIntegrate[P[i]gde[j],{x,-a/2,a/2},{y,-
b/2,b/2}],{i,1,nn},{j,1,nn}];
Do[F[[i]]=NIntegrate[P[i],{x,-a/2,a/2},{y,-
b/2,b/2}],{i,1,nn}];
cc=Table[C[i],{i,1,nn}];
sol=Solve[K.cc==F,cc];

(*Substitute the values of parameters c in w and evaluate the final w deflection equation and the related M, V, Q and R equations*)
```

```
w=Sum[C[i] P[i],{i,1,nn}]/.sol[[1]];
W[k,n]=w;
ν=3/10;
Mx[k,n]=-DD (D[w,{x,2}]+ν  D[w,{y,2}]);
My[k,n]=-DD ( D[w,{y,2}]+ν  D[w,{x,2}]);
Qx[k,n]=-DD ( D[D[w,x],{y,2}]+D[w,{x,3}]);
Qy[k,n]=-DD (D[D[w,y],{x,2}]+D[w,{y,3}]);
Vx[k,n]=-DD (D[w,{x,3}]+(2-ν) D[D[w,{y,2}],x]);
Vy[k,n]=-DD (D[w,{y,3}]+(2-ν) D[D[w,{x,2}],y]);
R[k,n]=2 DD (1-ν) D[ D[w,x],y],
{n,1,5}],{k,1,Length[α]}];

(*Using the above code, since n is from 1 to 5, then the problem is solved 5 times, in each time n increases and therefore increase the number of terms and the accuracy of the solution*)

(* now Mathematica is used to evaluate w, M, V, Q and R at any specified point on the plate*)
n=5;
Results=Table[{α[[k]],W[k,n]/.{x→ 0,y→ 0},Mx[k,n]/.{x→ 0,y→ 0},My[k,n]/.{x→ 0,y→ 0},W[k,n]/.{x→ 0,y→ α[[k]]/2},Mx[k,n]/.{x→ 0,y→ α[[k]]/2}},{k,1,Length[α]}];


TableForm[Results,TableHeadings->{None,{"b/a","(w)x=0,y=0","(Mx)x=0,y=0","(My)x=0,y=0","(w)x=0,y=b/2","(Mx)x=0,y=b/2"}}]

(* Table Result*)
```

b/a	$(w)_{x=0,y=0}$	$(M_x)_{x=0,y=0}$	$(M_y)_{x=0,y=0}$	$(w)_{x=0,y=b/2}$	$(M_x)_{x=0,y=b/2}$
0.5	0.0137146	0.124068	0.0116199	0.0146427	0.127338
0.75	0.0133415	0.123012	0.0208212	0.0148599	0.129674
1	0.013094	0.122635	0.0271145	0.0150077	0.130535
1.25	0.0129582	0.122554	0.0312491	0.0151048	0.13205
1.5	0.0129065	0.122913	0.0342298	0.0151605	0.132273
2	0.0128997	0.124648	0.0371419	0.0152248	0.134285

Appendix B

Following, is the general form of Mathematica code used in this research to derive rectangular plate deflection solutions based on the Ritz method.

```
Clear["Global`*"]
α={0.5,0.75,1,1.25,1.5,2};q=1;DD=1;a=1;
(*where α is the set of ratios b/a being studied, q is the
load function in x & y, a is the length of the side
parallel to the x axis and DD is the magnitude of the
flexural rigidity of the plate (D)*)

Do[Do[b=α[[k]];
(*where b is the length of the side parallel to the y
axis*)
   w= Sum[c[i,j]   x^i  y^j ,{i,0,n},{j,0,n}];
(*where here ϕ[i,j] is assumed to be  x^ i  y^ j,  w is the
assumed deflection equation and n sets the number of terms
used in the w function*)

(*Defining all the derivations of w*)
wx=D[w,x];wy=D[w,y];wxx=D[w,{x,2}];wyy=D[w,{y,2}];wxy=D[D[w
,x],y];wxxx=D[w,{x,3}];wyyy=D[w,{y,3}];wxxy=D[D[w,{x,2}],y]
;wyyx=D[D[w,{y,2}],x];mx=-DD(wxx+ν         wyy);my=-DD(wyy+ν
wxx);vy=-DD(wyyy+(2-ν) wxxy);vx=-DD(wxxx+(2-ν) wyyx);

(*Listing all the boundary conditions (depending on the
plate boundaries). Here, all boundary conditions are
applied, but also just applying essential boundary
conditions could be enough*)
```

```
eq1=w==0/.x→-a/2;
eq2=mx==0/.x→-a/2;
eq3=w==0/.x→a/2;
eq4=mx==0/.x→ a/2;
eq5=my==0/.y→-b/2;
eq6=vy==0/.y→-b/2;
eq7=my==0/.y→ b/2;
eq8=vy==0/.y→ b/2;

(*Applying SolveAlways function to make ϕ[i,j] satisfying all the boundary conditions*)
eqs={eq1,eq2,eq3,eq4,eq5,eq6,eq7,eq8};
sol=SolveAlways[eqs,{x,y}];
w=Simplify[w/.sol[[1]]];

(*Applying the Ritz method to solve for parameters c*)
coef=Cases[Cases[Variables[w],Except[x]],Except[y]];
nn=Length[coef];
w=Simplify[w/.Table[coef[[i]]□ C[i],{i,1,nn}]];
Do[P[i]=Coefficient[w,C[i]],{i,1,nn}];
Do[Lap[i]=Simplify[D[P[i],{x,2}]+D[P[i],{y,2}]],{i,1,nn}
];
Do[Har1[i]=Simplify[D[P[i],{x,2}]],{i,1,nn}];
Do[Har2[i]=Simplify[D[P[i],{y,2}]],{i,1,nn}];
Do[Har3[i]=Simplify[D[D[P[i],{x,1}],{y,1}]],{i,1,nn}];
K=Table[0,{i,1,nn},{j,1,nn}];
F=Table[0,{i,1,nn}];
Do[K[[i,j]]=NIntegrate[(Lap[i] Lap[j]-(1-□ )((Har1[i]
Har2[j]+ Har2[i]Har1[j])-2Har3[i] Har3[j])),{x,-
a/2,a/2},{y,-b/2,b/2}],{i,1,nn},{j,1,nn}];
Do[F[[i]]=NIntegrate[P[i],{x,-a/2,a/2},{y,-
b/2,b/2}],{i,1,nn}];
```

```
    cc=Table[C[i],{i,1,nn}];
    sol=Solve[K.cc==F,cc];

(*Substitute the values of parameters c in w and evaluate
the final w deflection equation and the related M, V, Q and
R equations*)
    w=Sum[C[i] P[i],{i,1,nn}]/.sol[[1]];
    W[k,n]=w;
    ν=3/10;
    Mx[k,n]=-DD (D[w,{x,2}]+ν  D[w,{y,2}]);
    My[k,n]=-DD ( D[w,{y,2}]+ν  D[w,{x,2}]);
    Qx[k,n]=-DD ( D[D[w,x],{y,2}]+D[w,{x,3}]);
    Qy[k,n]=-DD (D[D[w,y],{x,2}]+D[w,{y,3}]);
    Vx[k,n]=-DD (D[w,{x,3}]+(2-ν) D[D[w,{y,2}],x]);
    Vy[k,n]=-DD (D[w,{y,3}]+(2-ν) D[D[w,{x,2}],y]);
    R[k,n]=2 DD (1-ν) D[ D[w,x],y],
    {n,1,5}],{k,1,Length[α]}];

(*Using the above code, since n is from 1 to 5, then the
problem  has been solved 5 times, in each time n increass
and therefore increase the number of terms and the accuracy
of the solution*)

(* now Mathematica is used to evaluate w, M, V, Q and R at
any specified point on the plate*)
n=5;
Results=Table[{α[[k]],W[k,n]/.{x→ 0,y→ 0},Mx[k,n]/.{x→ 0,y→
0},My[k,n]/.{x→ 0,y→ 0},W[k,n]/.{x→ 0,y→
α[[k]]/2},Mx[k,n]/.{x→ 0,y→ α[[k]]/2}},{k,1,Length[α]}];
```

```
TableForm[Results,TableHeadings->{None,{"b/a","(w)x=0,y=0","(Mx)x=0,y=0","(My)x=0,y=0","(w)x=0,y=b/2","(Mx)x=0,y=b/2"}}]
```

```
(* Table Result*)
```

b/a	$(w)_{x=0,y=0}$	$(M_x)_{x=0,y=0}$	$(M_y)_{x=0,y=0}$	$(w)_{x=0,y=b/2}$	$(M_x)_{x=0,y=b/2}$
0.5	0.0137158	0.124222	0.0126599	0.0146381	0.127138
0.75	0.0133425	0.123106	0.0209665	0.0148626	0.130038
1	0.0130932	0.122469	0.0271694	0.0150097	0.130686
1.25	0.0129582	0.122462	0.0310977	0.0151148	0.131951
1.5	0.0128945	0.122329	0.0336383	0.0151564	0.132346
2	0.0128901	0.123858	0.0365491	0.0152003	0.132392

Appendix C

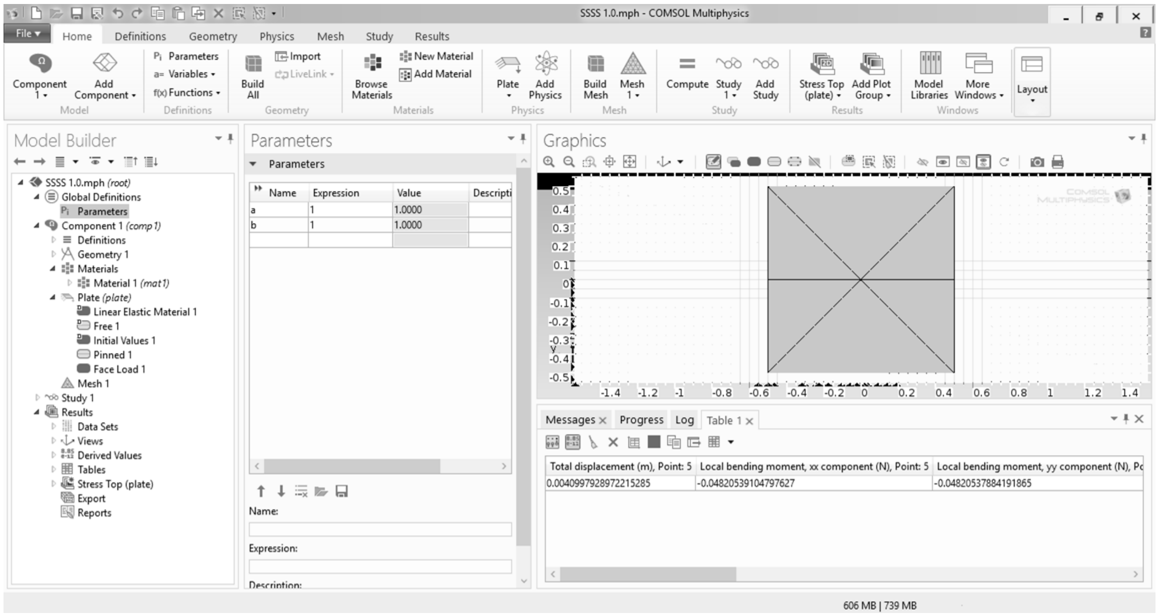

Figure 22 COMSOL Multiphysics User Interface

Figure 22 shows the general user interface of COMSOL Multiphysics software. Following are the main 10 steps used in this research to analyze plates using this software to get FEM solutions for plates:

1. Directly after the launch of COMSOL Multiphysics software, it asks if one wants to start with a file from the Model Wizard or a Blank Model, in our study the first option should be chosen.
2. After choosing the Model Wizard, the software asks to select the space dimension of our work and several options are given. Since our work is on plates which are analyzed as 2 dimension objects, 2D space is the proper choice.
3. Then, the software asks to select the physics. Under the option “Structural Mechanics”, there is “Plate” option, this option should be selected and then press

"add". Finally press "Study" to go to the next window which is shown in **Error! eference source not found.**. In this window, there is a small box with the title "Model Builder" on the left of the screen. From this small window all the parameters and details related to the plate case that we are analyzing can be entered. The following points go over the list in this window tap by tap.

4. The first tap in this box is "Global Definitions" and below this tap there is "Parameters" tap. In this tap all the parameters (constants or variables) that we want should be entered and give them some values. For our case, the parameters a and b are used which are the dimensions of the plates. In our work, a is always set to be 1 but b is changed for different values depending on the b/a ratio of the studied plate.
5. The next tap is "Component 1" which starts by the "Geometry" tap below it. In this tap the shape of the plate is defined (i.e. rectangle, ellipse. Etc.) as a function of the parameters defined in step 4. Any points or lines over the plate surface can be defined here also.
6. Below the "Geometry" there is "Material" tap. In this tap one can choose to define a new material. This is useful in this study because the used material should have D (the flexural rigidity of the plate) = 1 to be able to compare the results with the results derived using Mathematica which we use in it $D = 1$. COMSOL asks for three main properties of the material, which are: Young's Modulus (E), Poisson's Ratio (ν) and density (ρ). Since $D = \frac{Eh^3}{12(1-\nu^2)}$, then in most of the plate cases it is assumed that $h = 0.01$, $E = 10920000$ and $\nu = 0.3$ to get $D = 1$. Density (ρ) value has no effect so we set it to be zero.

7. Below the "Material" tap there is the "Plate" tap. Three main things are defined in this tap. The first thing is the depth d of the plate (or h) which is decided to be it 0.01. By write click on the "Plate" tap, more important things can be defined which are the boundary conditions and the load on the plate. Doing that is very easy, the type of boundary condition is selected and then the software will ask to specify the sides with this condition. The same thing for loads, the type of load is selected, the plate surface is chosen and then the magnitude of the load is entered. For the case of uniformly loaded plates, face load equal to -1 in z direction is used.
8. Below this tap is the "Mesh" tap. In this tap, the size of the analysis mesh is specified. The smaller the mesh, the better the results. After selecting the mesh size, the Build All button should be pressed.
9. The next tap is the "Study" tap. After selecting it and press the Compute button, the software will do all the analysis related to the plate.
10. After finishing the analysis, one moves to the next tap which is "Results" tap. From that tap, deflections, moments, shears, etc. at the previously defined points in "Geometry" tap can be evaluated or get some graphical 2D and 3D plots for any property over the surface of the plate. In this study the "Derived Values" inner tap was mainly used to set the wanted results to be evaluated at specific points and evaluate all of them to one table. The results got in these kind of tables are then compared with Mathematica results.

Appendix D

(A). Derived Galerkin method polynomial solutions for uniformly loaded fully simply supported rectangular plates with Poisson's ratio $v = 0.3$ & $b/a\ ratio = 1$ are listed here. Solutions are in the general form:

$$w = \left[\sum_{i=0}^{n}\sum_{j=0}^{n} C_{i,j}(a^2 - 4x^2)(b^2 - 4y^2)x^{2i}y^{2j}\right] * \frac{\boldsymbol{qa^4}}{\boldsymbol{D}}$$

where $-\frac{1}{2} \leq x \leq \frac{1}{2}$ & $-\frac{1}{2} \leq y \leq \frac{1}{2}$.

a. $n = 1$, $w = [1.655 * 10^{-4}(5 - 24x^2 + 16x^4)(5 - 24y^2 + 16y^4)\ ] * \frac{\boldsymbol{qa^4}}{\boldsymbol{D}}$

b. $\mathrm{n} = 2$, $w = [4.241 * 10^{-4}(-5 + 4x^2)(-1 + 4x^2)(-5 + 4y^2)(-1 + 4y^2)$
$-4.376 * 10^{-5}(-1 + 4x^2)(-1 + 4y^2)(180 - 320x^4 - 144y^2 + 256x^4y^2)$
$-1.751 * 10^{-4}(-5 + 4x^2)(-1 + 4x^2)(-1 + 4y^2)(-9 + 16y^4)$
$+1.138 * 10^{-4}(-1 + 4x^2)(-1 + 4y^2)(81 - 144x^4 - 144y^4 + 256x^4y^4)\ ] * \frac{qa^4}{D}$

c. $n = 3$, $w = [2.923 * 10^{-4}(-5 + 4x^2)(-1 + 4x^2)(-5 + 4y^2)(-1 + 4y^2)$
$-7.518 * 10^{-7}(-1 + 4x^2)(-1 + 4y^2)(2880 - 5120x^4 - 2304y^2 + 4096x^4y^2)$
$-2.343 * 10^{-6}(-1 + 4x^2)(-1 + 4y^2)(1040 - 5120x^6 - 832y^2 + 4096x^6y^2)$
$-4.812 * 10^{-5}(-5 + 4x^2)(-1 + 4x^2)(-1 + 4y^2)(-9 + 16y^4)$
$+2.239 * 10^{-5}(-1 + 4x^2)(-1 + 4y^2)(1296 - 2304x^4 - 2304y^4 + 4096x^4y^4)$
$-5.449 * 10^{-5}(-1 + 4x^2)(-1 + 4y^2)(468 - 2304x^6 - 832y^4 + 4096x^6y^4)$
$-2.343 * 10^{-6}(-1 + 4x^2)(-1 + 4y^2)(1040 - 832x^2 - 5120y^6 + 4096x^2y^6)$
$-5.449 * 10^{-5}(-1 + 4x^2)(-1 + 4y^2)(468 - 832x^4 - 2304y^6 + 4096x^4y^6)$
$+1.654 * 10^{-4}(-1 + 4x^2)(-1 + 4y^2)(169 - 832x^6 - 832y^6 + 4096x^6y^6)] * \frac{qa^4}{D}$

d. $n = 4$, $w = [5.034 * 10^{-3}(1-4x^2)(1-4y^2)(\frac{25}{16} - \frac{5x^2}{4} - \frac{5y^2}{4} + x^2y^2)$

$-4.544 * 10^{-3}(1-4x^2)(1-4y^2)\left(\frac{45}{64} - \frac{5x^4}{4} - \frac{9y^2}{16} + x^4y^2\right)$

$+1.904 * 10^{-3}(1-4x^2)(1-4y^2)\left(\frac{65}{256} - \frac{5x^6}{4} - \frac{13y^2}{64} + x^6y^2\right)$

$-2.935 * 10^{-2}(1-4x^2)(1-4y^2)\left(\frac{85}{1024} - \frac{5x^8}{4} - \frac{17y^2}{256} + x^8y^2\right)$

$-4.544 * 10^{-3}(1-4x^2)(1-4y^2)\left(\frac{45}{64} - \frac{9x^2}{16} - \frac{5y^4}{4} + x^2y^4\right)$

$+3.072 * 10^{-2}(1-4x^2)(1-4y^2)\left(\frac{81}{256} - \frac{9x^4}{16} - \frac{9y^4}{16} + x^4y^4\right)$

$-8.882 * 10^{-3}(1-4x^2)(1-4y^2)\left(\frac{117}{1024} - \frac{9x^6}{16} - \frac{13y^4}{64} + x^6y^4\right)$

$-1.136 * 10^{-1}(1-4x^2)(1-4y^2)\left(\frac{153}{4096} - \frac{9x^8}{16} - \frac{17y^4}{256} + x^8y^4\right)$

$+1.904 * 10^{-3}(1-4x^2)(1-4y^2)\left(\frac{65}{256} - \frac{13x^2}{64} - \frac{5y^6}{4} + x^2y^6\right)$

$-8.882 * 10^{-3}(1-4x^2)(1-4y^2)\left(\frac{117}{1024} - \frac{13x^4}{64} - \frac{9y^6}{16} + x^4y^6\right)$

$+8.449 * 10^{-1}(1-4x^2)(1-4y^2)\left(\frac{169}{4096} - \frac{13x^6}{64} - \frac{13y^6}{64} + x^6y^6\right)$

$-2.532(1-4x^2)(1-4y^2)\left(\frac{221}{16384} - \frac{13x^8}{64} - \frac{17y^6}{256} + x^8y^6\right)$

$-2.935 * 10^{-2}(1-4x^2)(1-4y^2)\left(\frac{85}{1024} - \frac{17x^2}{256} - \frac{5y^8}{4} + x^2y^8\right)$

$-1.136 * 10^{-1}(1-4x^2)(1-4y^2)\left(\frac{153}{4096} - \frac{17x^4}{256} - \frac{9y^8}{16} + x^4y^8\right)$

$-2.532(1-4x^2)(1-4y^2)\left(\frac{221}{16384} - \frac{17x^6}{256} - \frac{13y^8}{64} + x^6y^8\right)$

$+9.234(1-4x^2)(1-4y^2)(\frac{289}{65536} - \frac{17x^8}{256} - \frac{17y^8}{256} + x^8y^8)] * \frac{qa^4}{D}$

(B). Derived Galerkin polynomial solution for bending of uniformly loaded SSSC plate with b/a ratio $= 2$, $v = 0.3$ and $n = 4$.

$$
\begin{aligned}
w = &\left(\frac{qa^4}{D}\right)[2.305 * 10^{-3}(-5+4x^2)(-1+4x^2)(-1+y)^2(1+y)(2+y) \\
&+1.979 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(-288+512x^4-144y+256x^4y) \\
&-1.023 * 10^{-3}(-1+4x^2)(-1+y)^2(1+y)(-104+512x^6-52y+256x^6y) \\
&+7.214 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(-34+512x^8-17y+256x^8y) \\
&-3.674 * 10^{-3}(-5+4x^2)(-1+4x^2)(-1+y)^2(1+y)(-1+y+y^2) \\
&+9.763 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(144-256x^4-144y+256x^4y-144y^2 \\
&\qquad +256x^4y^2) \\
&-5.118 * 10^{-3}(-1+4x^2)(-1+y)^2(1+y)(52-256x^6-52y+256x^6y-52y^2+256x^6y^2) \\
&+8.506 * 10^{-3}(-1+4x^2)(-1+y)^2(1+y)(17-256x^8-17y+256x^8y-17y^2+256x^8y^2) \\
&-2.826 * 10^{-5}(-1+4x^2)(-1+y)^2(1+y)(-1600+1280x^2-640y+512x^2y-320y^2 \\
&\qquad +256x^2y^2-320y^3+256x^2y^3) \\
&-1.984 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(-720+1280x^4-288y+512x^4y-144y^2 \\
&\qquad +256x^4y^2-144y^3+256x^4y^3) \\
&+8.354 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(-260+1280x^6-104y+512x^6y-52y^2 \\
&\qquad +256x^6y^2-52y^3+256x^6y^3) \\
&-2.969 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(-85+1280x^8-34y+512x^8y-17y^2+256x^8y^2 \\
&\qquad -17y^3+256x^8y^3) \\
&+2.588 * 10^{-5}(-1+4x^2)(-1+y)^2(1+y)(960-768x^2-640y+512x^2y-640y^2 \\
&\qquad +512x^2y^2-320y^3+256x^2y^3-320y^4+256x^2y^4) \\
&-3.912 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(432-768x^4-288y+512x^4y-288y^2 \\
&\qquad +512x^4y^2-144y^3+256x^4y^3-144y^4+256x^4y^4) \\
&+2.084 * 10^{-3}(-1+4x^2)(-1+y)^2(1+y)(156-768x^6-104y+512x^6y-104y^2 \\
&\qquad +512x^6y^2-52y^3+256x^6y^3-52y^4+256x^6y^4) \\
&-3.482 * 10^{-3}(-1+4x^2)(-1+y)^2(1+y)(51-768x^8-34y+512x^8y-34y^2+512x^8y^2 \\
&\qquad -17y^3+256x^8y^3-17y^4+256x^8y^4) \\
&+1.301 * 10^{-5}(-1+4x^2)(-1+y)^2(1+y)(-2880+2304x^2-960y+768x^2y-640y^2 \\
&\qquad +512x^2y^2-640y^3+512x^2y^3-320y^4+256x^2y^4-320y^5+256x^2y^5) \\
&+4.501 * 10^{-5}(-1+4x^2)(-1+y)^2(1+y)(-1296+2304x^4-432y+768x^4y-288y^2 \\
&\qquad +512x^4y^2-288y^3+512x^4y^3-144y^4+256x^4y^4-144y^5+256x^4y^5) \\
&-1.115 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(-468+2304x^6-156y+768x^6y-104y^2 \\
&\qquad +512x^6y^2-104y^3+512x^6y^3-52y^4+256x^6y^4-52y^5+256x^6y^5) \\
&-2.113 * 10^{-4}(-1+4x^2)(-1+y)^2(1+y)(-153+2304x^8-51y+768x^8y-34y^2 \\
&\qquad +512x^8y^2-34y^3+512x^8y^3-17y^4+256x^8y^4-17y^5+256x^8y^5)]
\end{aligned}
$$

References

[1] S. Timoshenko, Theory of Plates and Shells, Mc-Graw Hill Book Company, 1959.

[2] C. Fo-van, "Bending of Uniformly CantileverRectangular Plates," Techmodern Business Promotion Centre, Hong Kong, 1980.

[3] R. L. Taylor and S. Govindjee, "Solution of Clamped Rectangular Plate Problems," University of California, Berkeley, Berkeley, CA, USA, 2002.

[4] C. İmrak and İ. Gerdemeli, "An Exact Solution for the Deflection of a Clamped Rectangular Plate under Uniform Load," Applied Mathematical Sciences, Istanbul, Turkey, 2007.

[5] M. Batista, "Uniformly Loaded Rectangular Thin Plates with Symmetrical Boundary Conditions," Univesity of Ljubljana, Slovenia, Jan 2010.

[6] C. SHUANG, SYMPLECTIC ELASTICITY APPROACH FOR EXACT BENDING SOLUTIONS OF RECTANGULAR THIN PLATES, Hong Kong: City University of Hong Kong, November 2007, p. 157.

[7] Y. Liu and R. Li, "Accurate bending analysis of rectangular plates with two adjacent edges free and the others clamped or simply supported based on new symplectic approach," Elsevier Inc., Liaoning Province, PR China, 2009.

[8] R. B. Bhaat, "PLATE DEFLECTIONS USING ORTHOGONAL POLYNOMIALS," Journal of Engineering Mechanics,, Montreal, Quebec, Canada, 1985.

[9] K. M. Liew, "RESPONSE OF PLATES OF ARBITRARY SHAPE SUBJECT TO STATIC LOADING," Journal of Engineering Mechanics,, Singapore, 1992.

[10] M. M. Saadatpour and M. Azhari, "The Galerkin method for static analysis of simply supported plates of general shape," Elsevier Science Ltd., Isfahan, Iran, 1998.

[11] C. F. Mbakogu and N. M. Pavlovic, "Bending of clamped orthotropic rectangular plates: a variational symbolic solution," Elsevier Science Ltd., London, UK, 1999.

[12] O. N. N and A. C. H, "Bending Analysis of Isotropic Rectangular Plate with All Edges Clamped: Variational Symbolic Solution," Journal of Emerging Trends in Engineering and Applied Sciences, Nsukka, Nigeria, 2011.

[13] A. Balasubramanian, "Plate Analysis with Different Geometries and Arbitrary Boundary Conditions," The University of Texas at Arlington, Texas, USA, December 2011.

[14] C. J. Ezeh, M. O. Ibearugbulem, H. E. Opara and O. A. Oguaghamba, "CHARACTERISTIC ORTHOGONAL POLYNIMIAL APPLICATION TO GALERKIN INDIRECT VARIATIONAL METHOD FOR SIMPLY SUPPORTED PLATE UNDER IN PLANE LOADING," International Journal of Research in Engineering and Technology, Owerri, Nigeria, April, 2014.

[15] O. M. Ibearugbulem, "Using the product of two mutually perpendicular truncated polynomial series as shape function for rectangular plate analysis," International Journal of Emerging Technologies and Engineering, Owerri, Nigeria, August 2014.

[16] F. O. Okafor and O. T. Udeh, "DIRECT METHOD OF ANALYSIS OF AN ISOTROPIC RECTANGULAR PLATE USING CHARACTERISTIC ORTHOGONAL POLYNOMIALS," Nigerian Journal of Technology, Nsukka, Nigeria, April, 2015.

[17] J. N. Reddy, Energy and Variational Methods in Applied Mechanics, Blacksburg, Virginia: Wiley Interscience, 1984.

[18] "WOLFRAM MATHEMATICA," Wolfram, 2016. [Online]. Available: https://www.wolfram.com/mathematica/. [Accessed August 2016].

[19] "COMSOL Multiphysics," COMSOL, 2016. [Online]. Available: https://www.comsol.com/comsol-multiphysics. [Accessed August 2016].

[20] A. Ramsay and E. Maunder, "An Error in Timoshenko's 'Theory of Plates and Shells'," *STRUCTURE Magazine,* pp. 28-32, July 2016.

[21] T. Q. Ye and Y. Liu, "Finite deflection analysis of elastic plate by the boundary element method," Xian, Shaanxi, China, 1984.

[22] X. Wang and Y. Wang, "Buckling analysis of thin rectangular plates under uniaxial or biaxial compressive point loads by the differential quadrature method," ElsevierLtd., 2015.

[23] A. C. Ugural, Stresses in Plates and Shells, 2nd Edition ed. ed., Singapore: McGraw-Hill, 1999.

[24] M. Tanaka, T. Matsumoto and Z.-D. Zheng, "Incremental analysis of finite deflection of elastic plates via boundary-domain-element method," Elsevier Inc., 1995.

[25] C. . R. Steele and C. D. Balch, "Introduction to the Theory of Plates," Stanford, 2009.

[26] S. Saad-Eldeen, Y. Garbatov and C. G. Soares, "Stress–strain analysis of dented rectangular plates subjected to uni-axial compressive loading," Elsevier Ltd., 2014.

[27] C. Roque, A. Ferreira and J. Reddy, "Analysis of Mindlin micro plates with a modified couple stress theory and a meshless method," 2012.

[28] C.-D. Chen, "Analytical solutions of stress singularities in a sectorial plate based on Reissner–Mindlin plate theory," Elsevier Inc., 2014.

Vitae

Name : Ahmed Abdulla Ahmed Ali Hasan Al-Ali

Nationality : Bahraini

Date of Birth : 10/3/1990

Email : ahmed_al_ali@outlook.com

Address : House 1438, Road 5526, Block 255, Galali, Bahrain

Academic Background:

King Fahd University of Petroleum & Minerals – Dhahran – KSA (2014-2016)

Master of Science degree in Civil Engineering (Structural Engineering) – first honor.

King Fahd University of Petroleum & Minerals – Dhahran – KSA (2008-2012)

Bachelor's degree first honor graduated student in Civil Engineering with a cumulative GPA of 3.959 out of 4.00.

Al-Hidaya Al-Khalifia Secondary School – Bahrain (2005-2008)

Graduated from Science Sector, holding GPA 98.8% - 5th highest GPA in Bahrain 2008.

.